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No such thing like perfect hammer: comparing different objective function specifications for optimal control

D. Blueschke* and I. Savin†

Abstract

The linear-quadratic (LQ) optimization is a close to standard technique in the optimal control framework. LQ is very well researched and there are many extensions for more sophisticated scenarios like nonlinear models. Usually, the quadratic objective function is taken as a prerequisite for calculating derivative-based solutions of optimal control problems. However, it is not clear whether this framework is so universal as it is considered. In particular, we address the question on whether the objective function specification and the corresponding penalties applied, are well suited in case of a large exogenous shock an economy can experience because of, e.g., the European debt crisis. While one can still efficiently minimize quadratic deviations in state and control variables around policy targets, the economy itself has to go through a period of turbulence with economic indicators, such as unemployment, inflation or public debt, changing considerably over time. In this study we test four alternative designs of the objective function: a least median of squares based approach, absolute deviations, cubic and quartic objective functions. The analysis is performed based on a small-scale model of the Austrian economy and finds that there is a certain trade-off between quickly finding optimal solution using the LQ technique (reaching defined policy targets) and accounting for alternative objectives, such as limiting volatility in the economic performance.

Keywords: Differential evolution; nonlinear optimization; optimal control; least median of squares; cubic optimization; quartic optimization

JEL Classification: C54, C61, E27, E61, E63.

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1 Introduction

Today, several countries in the European Union face difficulties in mitigating their public budget deficit and debt issues, which were triggered by the last economic crisis. In 2010, for example, the first bail-out program for Greece (of 110 Billion Euro) was approved by the Troika of the International Monetary Fund, European Central Bank and European Commission. In 2013, a 47.5% haircut for deposits above 100 Thousand Euro was applied for several Cypriot banks. Another bail-out program for Greece is under discussion right now.

For the Austrian economy (and the other countries of the Euro zone) such an event has a one-time negative impact on the budget balance. The question for the local government is how to react to such a budget balance shock. The optimal control framework is a well-known tool to address such a fiscal policy question. A 'traditional' way to consider optimal control problems is the linear quadratic (LQ) optimization technique. This technique is mainly based on works by Pontryagin *et al.* (1962) and Bellman (1957). There are several more sophisticated numerical algorithms based on the LQ optimization framework, which allow us to consider nonlinear problems as well, e.g., the OPTCON algorithm developed by Blueschke-Nikolaeva *et al.* (2012). However, the common for the LQ framework is its sensitivity to outliers. The objective function is formulated in quadratic way. A squared outlier influences the objective function considerably forcing an active use of control variables, which might be undesirable in certain situations. For example, Fatas and Mihov (2003) show that an aggressive use of fiscal policy induces significant macroeconomic instability. Thus, in case of a large exogenous shock, a policy maker faces an additional task of mitigating the effects of this shock without putting the stability of the whole system at risk.

In a recent study by Blueschke *et al.* (2013) a new way of handling optimal control problems is proposed. The authors test an evolutionary approach for this purpose, namely Differential Evolution (DE, Storn and Price (1997)), which does not rely on the LQ framework. The authors apply DE to optimal control problems in nonlinear dynamic economic systems with asymmetric objective function, where the 'traditional' OPTCON algorithm does not work. Application of the DE method increases computational time substantially but gives much more flexibility in designing the objective function and different system constraints.[1] In the present paper we aim to use this flexibility of the DE method by introducing and solving an optimal control problem with different specifications of the objective function. In particular, we test four alternative designs of the objective function: a least median of squares based approach (LMS), absolute deviations, cubic and quartic objective functions.

The least median of squares (LMS) estimator (Rousseeuw 1984) is among the best known robust estimators for linear problems. It has been widely used in numerous applications of finance and technology (see, for example, Winker *et al.* (2011)), and is considered a standard technique for the robust data analysis. The main advantage in minimizing the median of squares instead of mean is the robustness or rather non-sensitivity of the LMS framework to unique large outliers. We apply this framework to design the objective function in an optimal control problem and check how LMS behave in this case. The question which we address here is whether LMS style shaped objective function serves the goal of mitigating instability due to a one-time shock (which can be interpreted as

[1]The trade-off between flexibility and higher computational demand is well known for heuristic optimization. For a concise discussion of the matter see Gilli and Schumann (2014).

an outlier) or may this approach even increase the volatility of the resulting states and controls obtained by the optimal control exercise.

An alternative objective function specification may be in applying on deviations in states and controls a power different from two (power equal to two corresponds to the LQ framework). A simple intuition suggests that a power above two shall penalize any deviation stronger limiting volatility in states and controls generated by the exogenous shock. In particular, we consider the cubic and quartic penalties, which have not yet been addressed well in literature.[2] In addition, to get a more comprehensive understanding on how different penalties' exponents drive resulting optimal paths, absolute deviations (i.e. power equaling one) are also considered.[3]

The rest of the paper is structured as follows. In Section 2 we describe a model of the Austrian economy experiencing an exogenous shock and solve it using the well known LQ framework. Section 3 contains detailed description of alternative objective function specifications. Section 4 presents results of the comparative study. Section 5 concludes.

2 The ATOPT Model

In our study we consider a small nonlinear macroeconometric model of the Austrian economy (ATOPT). The ATOPT model can be seen as an extended Phillips curve connected with a simple model of the public finance sector. The model includes four state variables (inflation, unemployment rate, budget balance and public debt), one exogenous non-controlled variable and eight unknown (estimated) parameters. It includes one fiscal policy instrument, primary balance, which allows a policy maker to control the whole system. Furthermore, it includes a channel for an external shock acting on the budget balance. The annual data for the time periods 1987 to 2013 yield 36 observations.[4] The start period for the optimization is 2014 and the end period is 2023 (10 years).

Model equations:
(Standard deviations are given in parentheses)

$$pi_t = \underset{(0.27)}{-0.14} + \underset{(0.10)}{0.60} * pi_{t-1} + \underset{(1.67)}{5.48} * \frac{1}{ur_t}, \tag{1}$$

$$ur_t = \underset{(0.34)}{6.58} \underset{(0.04)}{-0.11} * gr\_exr_t + \underset{(0.16)}{0.72} * prim\_balance_t, \tag{2}$$

$$budget\_balance_t = \underset{(0.15)}{-2.65} + \underset{(0.11)}{0.69} * prim\_balance_t + bb\_shock_t, \tag{3}$$

$$debt_t = debt_{t-1} + budget\_balance_t. \tag{4}$$

Equations (1 - 2) can be regarded as an extended Phillips curve including a nonlinear influence of the unemployment rate (denoted by 'ur') on the inflation rate ('pi'). The unemployment rate[5] is mainly driven by exogenous indicators: the growth rate of

[2] To the best of our knowledge, the only exception is constituted by Bass and Webber (1966).

[3] See, for example, Luus *et al.* (2001) for the application of an absolute values based objective function in optimal control problems.

[4] The first three equations are estimated by OLS. The last equation is an identity.

[5] The unemployment rate can also be considered as a proxy for the economic situation and the output.

exports (‘gr_exr’)[6] and the fiscal policy of the national government. For the latter, the estimated model suggests an expansionary effect of the fiscal policy. The primary balance is the fiscal policy instrument, which is under direct control of the Austrian government. In contrast, the budget deficit (or surplus if positive) (denoted by ‘$budget\_balance$’) is estimated based on primary balance as stated in equation (3). Furthermore, the budget balance can be influenced directly by an exogenous shock (‘bb_shock_t’). In our study, we apply a negative shock, which increases the budget deficit. Finally, the changes in the public debt level (‘$debt$’) are driven by the budget deficits (or surpluses if positive), as given in equation (4).

In the present study, the exogenous shock is modeled *ex-ante* as if the government knew what a budget balance shock it would face in few years time due to the European debt crisis. This is done mainly for simplicity to speed up the calculation and to not concentrate on the explanatory part of the model given its limitations.[7]

The ATOPT model, as stated in equations (1 - 4), captures a highly aggregated dynamics of the Austrian economy. We are aware that this is not sufficient to get accurate insights into the economic and/or fiscal situation in Austria. Instead, we use this model to test the performance of proposed approaches in case of a one-time shock, which increases the budget deficit. The initial values, the target values and the weights of the variables considered in the objective function are reported in Table 1.

Table 1: Objective variables in the ATOPT model

variable	initial value	target value	weight
pi	1.6	2	1
ur	7.6	6	1
$budget\_balance$	-1.5	0	1
$debt$	74.5	74.5$\searrow$60	0.2
$prim\_balance$	0.7	0	1

Note: The symbol $\searrow$ indicates that the target values for the objective variable *debt* are calculated in a linear decreasing way starting at initial value 74.5 and reaching the value 60 at the end of the planning horizon.

2.1 Nonlinear quadratic optimal control

In the first step we consider a standard optimum control problem with a quadratic objective function (a loss function to be minimized) and a nonlinear multivariate discrete-time dynamic system. The inter-temporal objective function is formulated in quadratic tracking form, which is often used in applications of optimal control theory to econometric models. ‘Traditionally’ it can be written as

$$J = \sum_{t=1}^{T} L_t(x_t, u_t) \tag{5}$$

[6] Which is a reasonable assumption for the small, trade-dependent Austrian economy.

[7] However, the exercise can be easily extended to model an unexpected shock. For this reason, one would have to apply the DE algorithm twice: first, before the shock is known on the entire period of interest; and second; once shock takes place - at the period between the shock and the remaining part of the calculation period. Again, given that the focus of the present paper is the specification of the objective function to be applied and not the economic conclusion the ATOPT model provides, we choose a simpler ex-ante option.

with

$$L_t(x_t, u_t) = \frac{1}{2} \begin{pmatrix} x_t - \tilde{x}_t \\ u_t - \tilde{u}_t \end{pmatrix}' W_t \begin{pmatrix} x_t - \tilde{x}_t \\ u_t - \tilde{u}_t \end{pmatrix}, \tag{6}$$

where x_t is an n-dimensional vector of state variables that describes the state of the economic system at any point in time t, u_t is an m-dimensional vector of control variables, $\tilde{x}_t \in R^n$ and $\tilde{u}_t \in R^m$ are given 'ideal' (target) levels of the state and control variables, respectively. T denotes the terminal time period of the finite planning horizon. W_t is an $((n+m) \times (n+m))$ matrix specifying the relative weights of the state and control variables in the objective function. The W_t matrix may also include a discount factor α, $W_t = \alpha^{t-1}W$. W_t (or W) is symmetric. The dynamics of the system is given by ATOPT model as stated in equations (1 - 4).

In order to solve the stated nonlinear optimal control problem, the OPTCON algorithm (Blueschke-Nikolaeva *et al.* 2012) is used. This algorithm allows for a numerical approximation of a nonlinear solution based on the standard techniques of linear quadratic optimization (LQ).

2.2 Optimal quadratic control of the ATOPT model in case of a one-time shock

We apply the nonlinear quadratic framework as stated in previous section to calculate the optimal fiscal policy for the Austrian economy in presence of an external one-time shock on budget balance. The start period for the optimization is 2014 and the end period is 2023. We assume the shock to occur in 2016. As we model a negative shock, we set the variable *bb_shock* (see equation (3) defining the government budget balance) to be -7 in period 3, i.e. $bb\_shock_3 = -7$. Thus, it is assumed that due to the exogenous shock the budget balance of the Austrian economy worsens by 7 percentage points in 2016.

We present here the optimal solution in situations with and without external shock, in order to show the effects of such a negative event to the Austrian economy. Figure 1 shows the optimal path for the control variable (primary balance), while Figure 2 presents the optimal paths for the state variables (the rate of inflation, the unemployment rate, the government budget balance and the public debt).

Graphical results show a strong trade-off between fiscal stability and output oriented policy, which goes in line with the 'philosophy' of the ATOPT model. In the experiment without shock, the LQ approach requires to run a restrictive fiscal policy in order to stabilize financial situation. The policy is required to be more active at the beginning of the planning horizon ($prim\_balance_1 = 4.9$), continuously decreasing to the end ($prim\_balance_{10} = 1.8$).

In the scenario with the shock, the LQ approach requires to run an even more restrictive fiscal policy in the 'shocked' period trying to compensate the negative impact of the shock ($prim\_balance_3 = 8.3$ and $budget\_balance_3 = -3.9$ vs. $bb\_shock_3 = -7$). In order to measure the volatility of such a discretionary policy, we calculate a weighted variance ($WVar$) of the time series involved. In particular, after calculating variances of each specific variables (both the control and the states), we aggregate them into one indicator using the weights applied in the earlier stage of optimization. This has an advantage of differently accounting for the variables (given that they have different units of measurement and, as a result, different order of variance involved), but at the same time not using any other arbitrary weights increasing the number of parameters affecting the

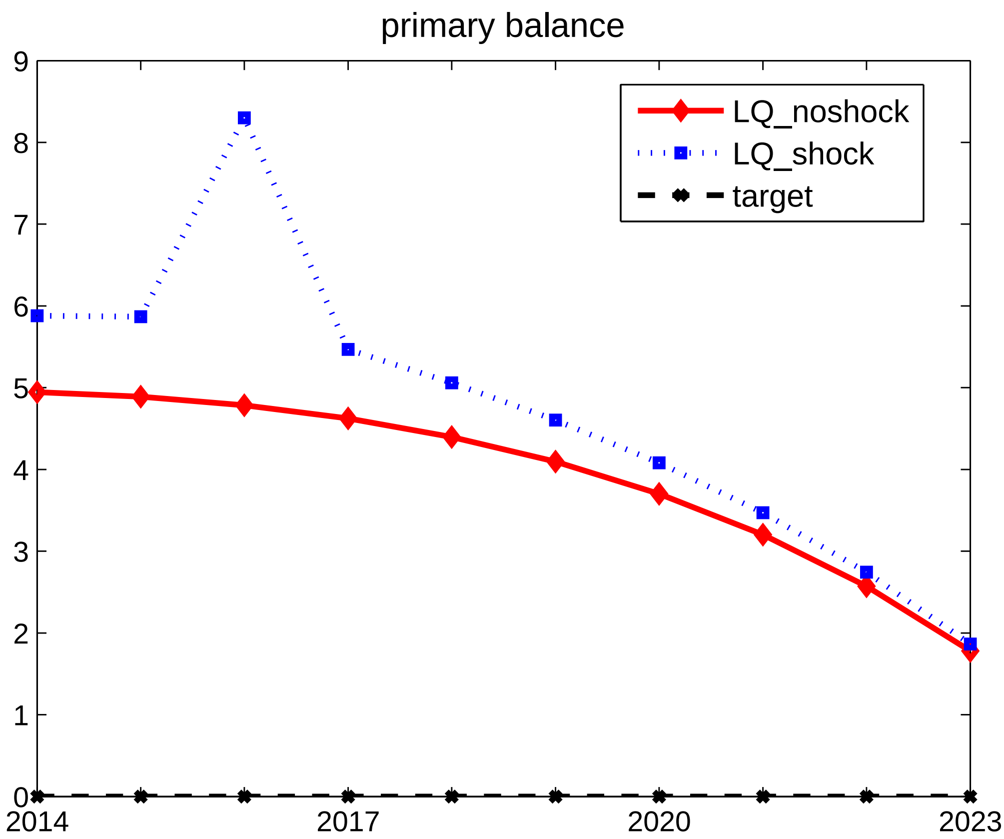

Figure 1: Control variable (LQ solution)

results.[8]

$WVar$ in the non-shocked scenario for the LQ framework is 22.71 and rises to 72.50 in the shocked scenario. The LQ framework is more or less forced to require a very active fiscal policy, due to quadratic costs of the outlier event. Such an intensive and restrictive fiscal policy has a relatively strong negative impact on the economic situation with the unemployment rate rising by more than two percentage points in one period.

In the next section we test alternative forms of objective function specification. A special attention should be paid to the excess volatility of fiscal policy.

3 Different Objective Function Specifications

In this section, we describe four alternative objective function specifications for optimal control framework. The proposed alternatives include an experiment using the idea of least median of squares (LMS), absolute deviations (ABS), a cubic objective function (CUB) and a quartic objective function (QUART).

[8] An alternative would have been to measure the coefficient of variation (CV), which standardizes the variances by the corresponding averages making it easy to compare between different variables and aggregate them into one single indicator. However, given that some of our variables have averages close to zero, this biases the result making CV not applicable.

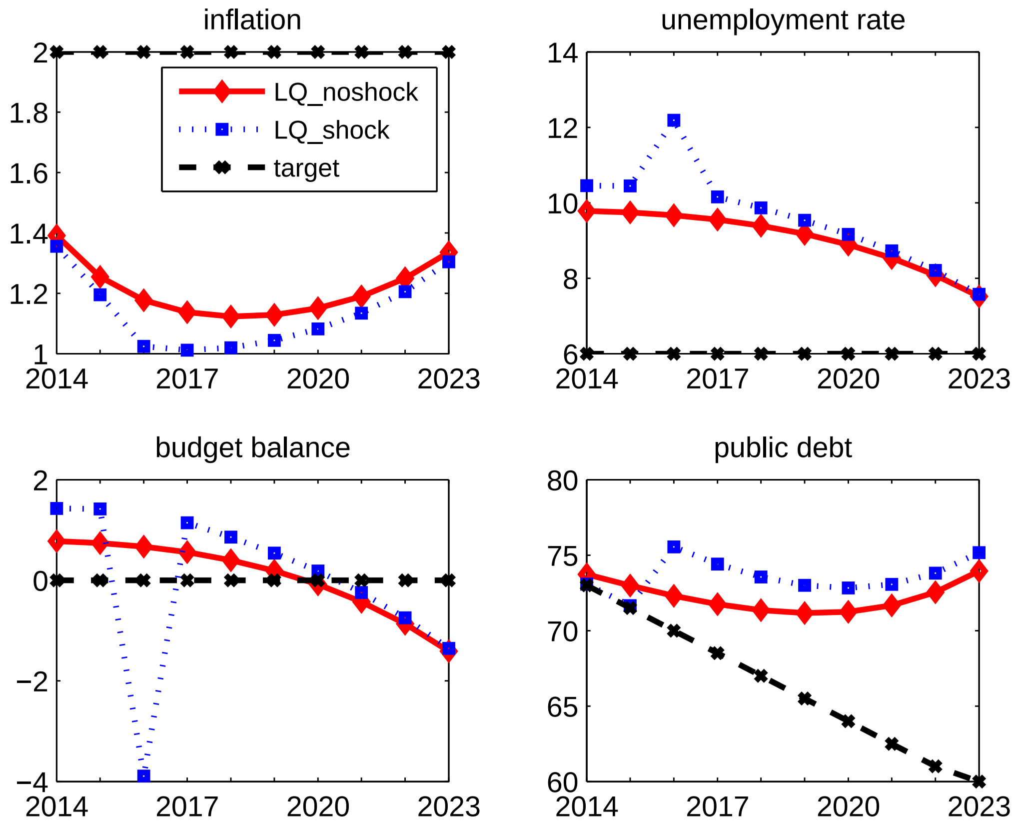

Figure 2: State variables for ATOPT (LQ solution)

Least median of squares

We reformulate the objective function 'J' (equations (5)-(6)) using the median of squares instead of the sum of squares on the corresponding states but not the control(s).[9] The intuition behind using a robust method like LMS is that it will devote little attention to the external shock taking place making the framework more robust to external effects. In particular, one can expect the volatility of the optimal paths of the states and controls be lower using LMS than without it. As a result we get the following objective function:

$$J = \sum_{i=1}^{N} median(L_1^{x_i}, L_2^{x_i}, ..., L_T^{x_i}) \cdot T + \sum_{t=1}^{T} L_t^u, \tag{7}$$

$$L_t^x(x_t) = \frac{1}{2}\left(x_t - \tilde{x}_t\right)' W_t^x \left(x_t - \tilde{x}_t\right), \tag{8}$$

$$L_t^u(u_t) = \frac{1}{2}\left(u_t - \tilde{u}_t\right)' W_t^u \left(u_t - \tilde{u}_t\right). \tag{9}$$

[9]We also have tested the possibility in applying the median-of-squares both on the states and controls, but the resulting dynamics of the variables under consideration becomes even more volatile and intractable in such a case and we do not proceed in considering it further. Results are available on request.

L_t^x represents the squared deviations between the state variables and their target values. L_t^u represents the squared deviations between the control variables and their target values. As stated above and given in equation (7), the objective function for states is calculated as a median (over time) of squares (corresponds to the LMS approach). The control variables are handled in a traditional way in the objective function by summarizing the squares (corresponds to LQ framework). The difference in levels between L^x and L^u needs to be adjusted by factor T as given in equation (7).

Absolute values

We calculate the objective function based on normal (non-quadratic or rather power equaling to one) deviations. In order to prevent the problem of offsetting positive and negative numbers, the absolute values of the calculated deviations are used.

$$J = \sum_{t=1}^{T} L_t^x + \sum_{t=1}^{T} L_t^u, \tag{10}$$

$$L_t^x(x_t) = \frac{1}{2} \left| x_t - \tilde{x}_t \right|' W_t^x, \tag{11}$$

$$L_t^u(u_t) = \frac{1}{2} \left| u_t - \tilde{u}_t \right|' W_t^u. \tag{12}$$

Cubic objective function

In this scenario the deviations from the targets are penalized by factor three. Similar to the previous scenario, we use the absolute values of deviations before calculating the exponent in order to prevent the problem of positive and negative numbers.

$$J = \sum_{t=1}^{T} L_t(x_t, u_t), \tag{13}$$

with

$$L_t(x_t, u_t) = \frac{1}{2} \left(\begin{pmatrix} x_t - \tilde{x}_t \\ u_t - \tilde{u}_t \end{pmatrix}' \right)^{1.5} W_t \left(\begin{pmatrix} x_t - \tilde{x}_t \\ u_t - \tilde{u}_t \end{pmatrix} \right)^{1.5}. \tag{14}$$

Quartic objective function

Finally, we calculate the deviations between state variables and the corresponding target paths to the power four.

$$J = \sum_{t=1}^{T} L_t(x_t, u_t), \tag{15}$$

with

$$L_t(x_t, u_t) = \frac{1}{2} \left(\begin{pmatrix} x_t - \tilde{x}_t \\ u_t - \tilde{u}_t \end{pmatrix}' \right)^{2} W_t \left(\begin{pmatrix} x_t - \tilde{x}_t \\ u_t - \tilde{u}_t \end{pmatrix} \right)^{2}. \tag{16}$$

4 Optimal Control Results

The four presented objective function alternatives and the nonlinear dynamic system given by the ATOPT model constitute four different optimal control problems. The stated problems are solved using the Differential Evolution method as proposed by Blueschke *et al.* (2013).[10]

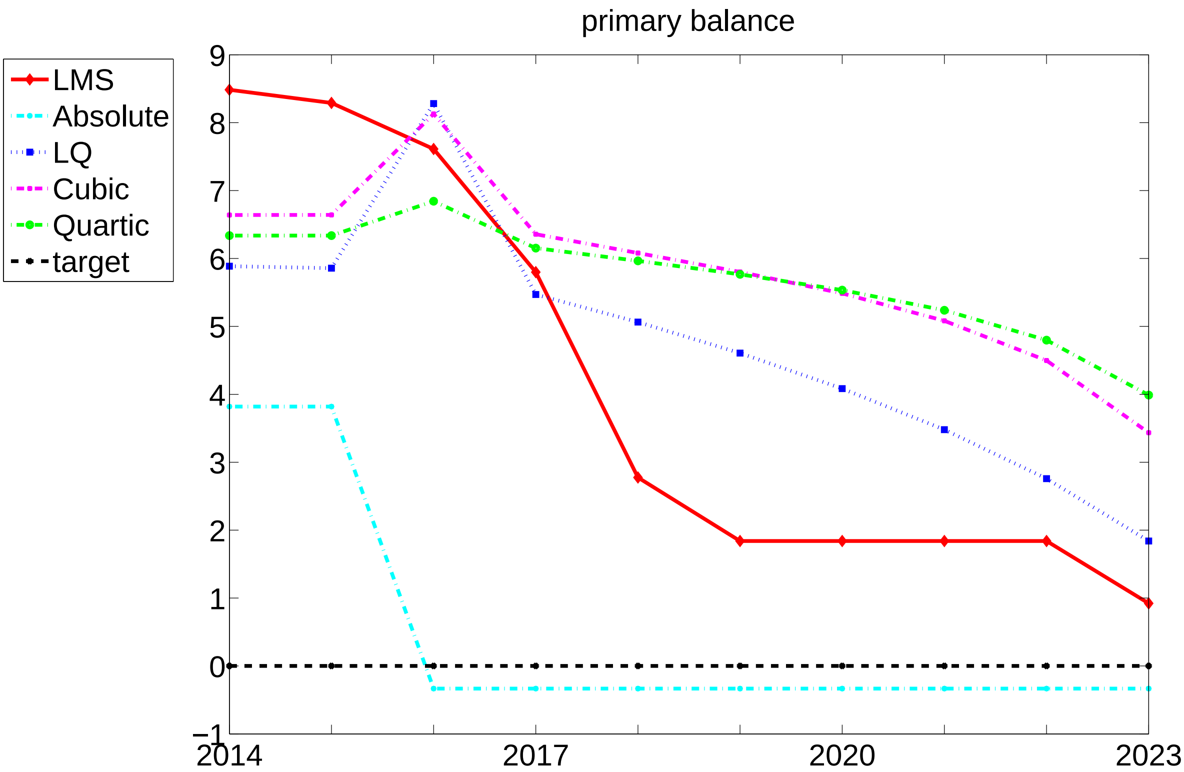

Figure 3: Optimal paths of control variable

Figures 3 and 4 show the optimal results for the four proposed objective function specifications in presence of a one-time shock. The LQ approach requires to run an extremely restrictive fiscal policy in the third period (2016) trying to compensate the negative impact of the shock. The LMS approach allows to smooth the effects of the one-time shock but increases the overall volatility and significantly differs from the optimal path both in control and state variables. Thus, the robust characteristics of the original LMS approach do not hold for optimal control problems.

The ABS scenario produces results dramatically deviating from other alternatives. Considering the absolute deviations in the objective functions allows one to give much less importance to the one-time shock but at the same time it does ignore larger deviations in

[10] It is worth mentioning that we conducted a series of simulation experiments to tune DE parameters, as described in Blueschke *et al.* (2013, p. 825-826), and as a result set for the LMS objective function specifications: the scale factor $F = 0.5$, the crossover rate $CR = 0.8$, the population size $p = 50 \times m \times T$ (where m is the number of controls and T is the number of time periods involved), while the number of DE generations $g^{max} = 2500$. These parameters are taken sufficiently large to ensure convergence. For the remaining objective function specifications being apparently simpler to solve, we set the following parameters: $F = 0.4$, $CR = 0.1$, $p = 10 \times m \times T$, while $g^{max} = 750$. For each objective function DE is restarted ten times.

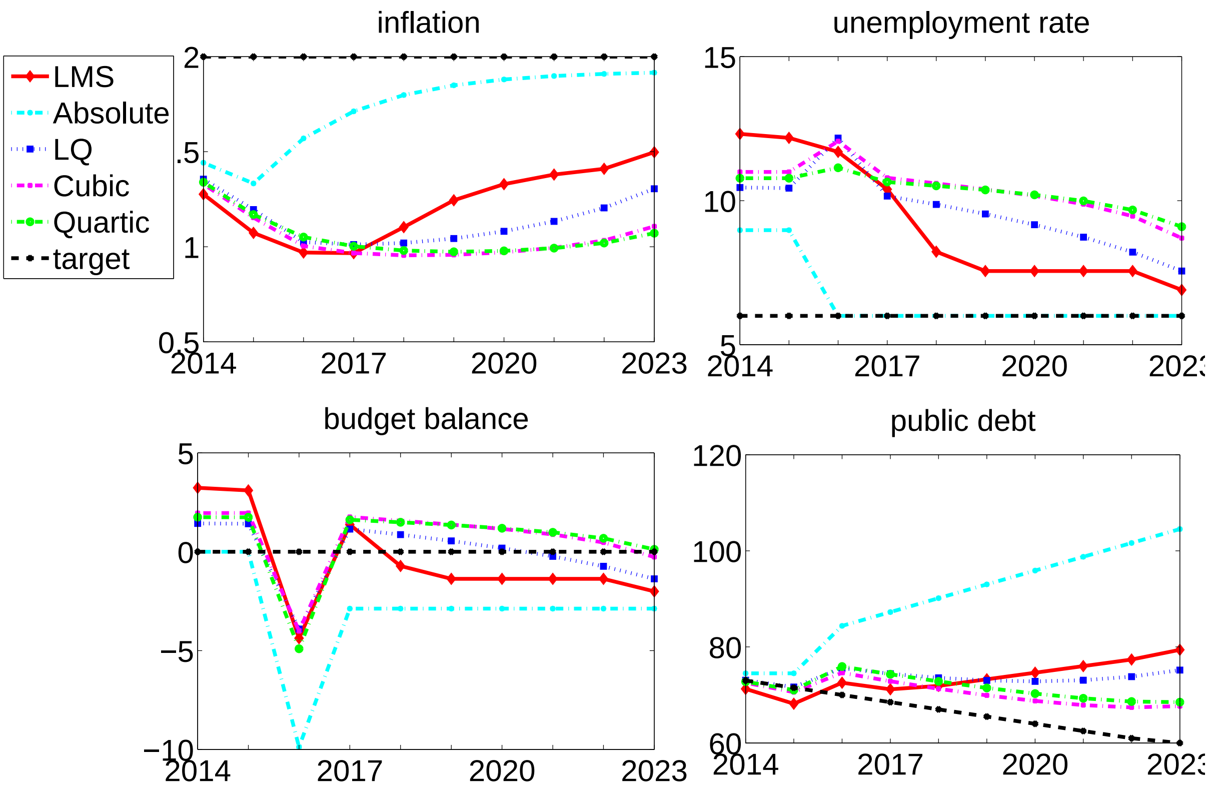

Figure 4: Optimal paths of state variables

states as well. In such a case, the government is required to run an extremely expansionary fiscal policy with strong positive effects for the unemployment but drastic consequences for the public finance with the public debt exceeding 100% of GDP at the end of the planning horizon.

The cubic and the quartic objective functions, in contrast, restrict the volatility of the optimal paths of the corresponding states and controls, thus, constituting a more robust way a policy maker can respond to an exogenous shock. The reason is that the penalty rises exponentially for any deviation from the targets stated so that the effects of the shock are absorbed by a larger number of periods. Clearly, the larger the exponent of the penalty, the smoother the paths of the corresponding variables.

Table 2 summarizes the results for the LQ framework and its four alternatives in terms of:

- minimal objective function values J achieved. These values are estimated using different function specifications and respective penalties applied making them incomparable between each other;
- standard deviation in DE results over restarts and cpu time required to obtain the results (per restart);
- minimal objective function values estimated with quadratic penalties (i.e. applying LQ framework to the states and controls achieved with different objective functions) LQJ. This presents a LQ-normalized and, thus, more comparable basis for the functions but also – as expected – indicates LQ results to be the best;

- the above-mentioned weighted variance ($WVar$), which allows to compare the objective functions in terms of the volatility in state and control time series.

First column in Table 2 contains the objective function values for the uncontrolled solution which uses the initial states of the corresponding systems and is aimed only for comparison with the optimal results. Second column contains optimal objective values as calculated by the OPTCON2 algorithm. Fifth column gives the objective values as calculated by Differential Evolution (DE) using a standard objective function (LQ framework) demonstrating that DE converges to basically the same solution as OPTCON but taking considerably more time ('cpu'). Third column gives the objective values as calculated by DE using LMS objective function (LMS), while fourth column contains the results for the objective function with absolute values (ABS). The last two columns state the results for cubic (CUB) and quartic (QUART) objective functions.

In terms of the LQ-normalized objective function values (LQJ) all alternatives except ABS show more or less similar results. Certainly, the scenario with LQ function to be minimized demonstrates the best result, but this simply indicates that the DE algorithm does its work well. If one would have normalized the results not on the quadratic but, e.g., on the quartic objective function, there is no doubt, the QUART result would have been the lowest. However, taking the volatility (WVar) into account there are significant differences in performance. While higher order of power in the objective function leads to a reduction of the volatility, the LMS and the ABS approaches results in much higher instability of the Austrian economy.

Hence, the actual trade-off one is facing is to find an optimum within the shortest computational time (second column in Table 2) or use a more sophisticated objective function accounting for additional policy objectives. In this particular case, the alternatives applying on deviations from the targets a power above two seem most promising. The larger the power, the smaller the variance in states and controls over time. Thus, we find that using larger exponents on deviations in states and controls one can better restrict volatility in macroeconomic variables while ensuring that the system reacts to the external shocks and minimizes its deviations from the targets given.

5 Conclusions and Outlook

In this paper we compare alternative forms of objective function specifications in the context of nonlinear dynamic optimal control problems experiencing a one-time exogenous shock. Given that the alternative specifications are not necessarily well-behaved, we optimize them not by means of linear-quadratic optimization technique but using an evolutionary algorithm, namely Differential Evolution.

Applying the alternative functional forms on a new small-scale model of the Austrian economy (ATOPT), we find that the traditional (quadratic) objective function performs not as good as its alternatives (using a power above two) in restricting volatility of the resulting optimal paths of the corresponding states and controls. The latter, in fact, has an important side effect on macroeconomic instability, which a policy maker tends to avoid. This serves as a first evidence that the (historical) dominance of the quadratic tracking form objective function specification present in literature, due to its convenience in applying LQ framework in finding its optimum, may not be an optimal choice for various problems and model scenarios to be considered. Hence, a more thorough selection

Table 2: Results for the ATOPT model with different settings

		OPTCON2		Differential Evolution				
		uncontr	optimal	LMS	ABS	LQ	CUB	QUART
no shock	J	693.20	188.91	126.36	73.75	188.91	2021.34	4437.39
	std	n/a	n/a	(0.0000)	(0.0000)	(0.0003)	(0.0006)	(0.0009)
	cpu	.001s	.5s	487s	472s	99s	310s	284s
	LQJ	n/a	188.91	240.39	547.45	188.91	212.61	208.92
	WVar	n/a	22.71	138.62	178.55	22.71	18.37	11.54
with shock	J	1076.56	291.43	207.25	91.95	291.43	3760.97	9776.78
	std	n/a	n/a	(0.0000)	(0.0000)	(0.0007)	(0.0010)	(0.0022)
	cpu	.001s	.5s	665s	521s	116s	316s	315s
	LQJ	n/a	291.43	334.71	884.78	291.43	321.96	320.05
	WVar	n/a	72.50	198.52	304.38	72.50	62.90	56.73

Note: Results for LQJ (linear-quadratic J) for alternative objective function specifications are obtained by re-evaluating the identified optimal sets of states and controls with standard LQ function. Results on $WVar$ (weighted variance) are calculated by estimating variance in the distribution of the states and controls obtained and accounting for the weights of the variables in the optimal control exercise as given in Table 1.

of a suitable objective function accounting for multiple objectives pursued by policy maker is necessary.

To sum up, the study demonstrates that other functional forms of an objective function can be easily applied when using the DE method for solving optimal control problems. Moreover, it is illustrated that the quadratic functional form is not a 'perfect hammer' accounting for different objectives one has. Formulation of more exact functional forms better addressing specific objectives one is willing to balance remains problem-specific and is left for further research.

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