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The Minority of Three-Game: An Experimental and Theoretical Analysis*

Thorsten Chmura, Werner Güth, Thomas Pitz and Anthony Ziegelmeyer[†]

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Abstract

We report experimental and theoretical results on the minority of three-game where three players have to choose one of two alternatives independently and the most rewarding alternative is the one chosen by a single player. This coordination game has many asymmetric equilibria in pure strategies that are non strict and payoff-asymmetric, and a unique symmetric mixed strategy equilibrium in which each player's behavior is based on the toss of a fair coin. We show that such a straightforward behavior is predicted by Harsanyi and Selten's (1988) equilibrium selection theory as well as alternative solution concepts like impulse balance equilibrium and sampling equilibrium. Our results indicate that participants rely on various decision rules, and that only a quarter of them decide according to the toss of a fair coin. Reinforcement learning is the most successful decision rule as it describes best the behavior of about a third of our participants.

KEYWORDS: Coordination; Minority game; Mixed strategy; Learning models; Experiments.

JEL CLASSIFICATION: C72; C91; D83.

1 Introduction

Experimental paradigms like the Prisoners' Dilemma, Public Goods Games, the Dictator Game, the Ultimatum Game or the Trust Game are game-theoretically trivial when players are selfish (and this is known). They either require no strategic reasoning at all, or the anticipation of others' rationality. Hence, game theory offers a clear-cut and testable prediction.¹

However, many games have multiple (perfect) equilibria which may question an equilibrium as a potentially satisfactory description of how a game will be played. One such class of games are market entry games (Selten and Güth, 1982) which capture the typical coordination problem when a newly emergent profit opportunity can be exploited only by a limited number of agents. In a market entry game each player enters one of several markets. For at least one of the markets, the payoff when entering that market decreases in the number of entrants. Market entry games thus share two essential characteristics: First, players have a common interest in selecting different actions, and, second, players face the same set of choices and similar incentives. Due to their common interest in selecting different actions, players would like to have some external clues to determine how different players should act but, especially in the case of symmetry, the game does not offer any such clues when more than two

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¹These paradigms have been subject to a very large number of experimental tests by both psychologists and economists because the game-theoretical prediction fits poorly the experimental data (among others, see Ledyard, 1995 and Camerer, 2003, chapter 2 for surveys). Needless to say, evidence of mutual cooperation in the Prisoners' Dilemma, public goods contribution, dictator allocation, ultimatum rejection, and trust repayment does *not* falsify game theory, per se. In experiments, games are usually played in money. Consequently, the mentioned experimental paradigms are tests of a joint hypothesis of game-theoretic behavior coupled with some assumption about utilities over money outcomes.

players are involved. There is typically a large class of asymmetric equilibria in pure strategies, each of them maximizing joint payoffs, and a unique inefficient mixed strategy equilibrium. Market entry game experiments have shown that behavior is consistent with reinforcement learning and that information about others' choices shapes the behavioral adjustment over time (see Ochs, 1999 and Camerer, 2003, chapter 7, section 3 for reviews).

In this paper, we present experimental evidence on the minority of three-game which differs in one important dimension from the previously studied market entry games: All its asymmetric equilibria in pure strategies are non strict and imply a payoff-asymmetry between the two resulting parties. Consequently, the unique symmetric mixed strategy equilibrium seems to be the natural benchmark to which we can compare observed behavior. According to the unique symmetry invariant equilibrium of the minority of three-game each player's behavior is based on the toss of a fair coin. Such a straightforward behavior is also predicted by alternative solution concepts like impulse balance equilibrium and sampling equilibrium. Thus, deviations from the mixed strategy hypothesis results most likely from heterogeneous decision rules.

Our experimental setup provides an adequate environment to identify such alternative decision rules. First, we endow experimental subjects with a mixing device to directly elicit mixed strategies and to allow subjects to generate i.i.d sequences of choices. Indeed, unlike in games with many interacting parties where (population) shares of different strategies may be interpreted as a mixed population strategy, triadic interaction is better studied by directly eliciting individual mixing. Of course, the latter might result from strategic uncertainty, in the sense of ambiguous expectations, rather than from indifference. Since we control for information retrieval, we hope to disentangle the hypotheses of strategic uncertainty (to be correlated with more retrievals) and of indifference (previous choices of others render both choices equally good). Second, we implement a *strangers design* which increases the difficulty to adapt to others' past play. Such changes may, of course, question the findings of former experiments employing market entry games like convergence to equilibrium play via reinforcement learning and the effects of information feedback. In this sense, our study appears like a stress test of how robust the former findings are.

In Section 2 we provide a thorough theoretical analysis of the minority of three-game. Section 3 describes the experimental protocol and Section 4 presents the experimental results. Section 5 discusses previous related research and Section 6 concludes.

2 Theory

In this section, we first introduce the minority of three-game and we derive its standard game-theoretical predictions. Second, we theoretically analyze the uniformly and an asymmetrically perturbed version of the game, and we discuss the implications of Harsanyi and Selten's (1988) equilibrium selection theory. Finally, we derive alternative predictions for the minority of three-game. Proofs can be found in the appendix.

2.1 The minority of three-game

Three players have to choose one of two alternatives independently and the most rewarding alternative is the one chosen by a single player. Hence, the two alternatives are perfectly symmetric and players' payoffs are solely based on how players distribute between them. Formally, we denote the minority of three-game by $M3G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where $N = \{1, 2, 3\}$ is the set of players, $A_i = \{X_i, Y_i\}$ for each $i \in N$ is the set of alternatives, and $u_i : \mathcal{A} \rightarrow \mathbb{R}$ with $\mathcal{A} = \times_{i \in N} A_i$ is player i 's (vNM) utility function such that

$$u_i(a_i, a_{-i}) = \begin{cases} 1 & \text{if } a_i \neq a_j \text{ and } a_i \neq a_k \\ 0 & \text{otherwise,} \end{cases}$$

where $i, j, k \in N$ with $i \neq j, i \neq k$ and $j \neq k$, $a_i \in A_i$ and $a_{-i} \in \times_{j \neq i} A_j$.² The normal-form representation of the minority of three-game is given by Table 1 where the first (respectively the second and third) element in a payoff vector corresponds to player 1's payoff (respectively player 2's payoff and player 3's payoff). As usual, we denote by $\Delta(A_i)$ the set of probability distributions over A_i and we refer to $\sigma_i \in \Delta(A_i)$ as a mixed strategy of player $i \in N$. The mixed extension of $M3G$ is $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where $U_i : \times_{i \in N} \Delta(A_i) \rightarrow \mathbb{R}$ is such that $U_i(\sigma) = \sum_{a \in \mathcal{A}} (\prod_{i \in N} \sigma_i(a_i)) u_i(a)$ for each $\sigma \in \times_{i \in N} \Delta(A_i)$.

		Player 3	
		X_3	Y_3
		Player 2	
		X_2	Y_2
Player 1	X_1	0, 0, 0	0, 1, 0
	Y_1	1, 0, 0	0, 0, 1

		Player 2	
		X_2	Y_2
Player 1	X_1	0, 0, 1	1, 0, 0
	Y_1	0, 1, 0	0, 0, 0

Table 1: The minority of three-game.

There exists 6 pure strategy equilibria: (X_1, Y_2, X_3) , (X_1, X_2, Y_3) , (X_1, Y_2, Y_3) , (Y_1, X_2, X_3) , (Y_1, Y_2, X_3) , and (Y_1, X_2, Y_3) . These pure strategy equilibria are Pareto efficient and non-strict since each of the two players with 0-payoff can deviate unilaterally without affecting her own payoff. Actually, the best reply structure of the game is rather simple since each player $i \in N$ should choose alternative X_i (resp. Y_i), if the sum of the probabilities for alternative X (resp. Y) by her opponents is strictly lower than 1. Indeed, for each player $i \in N$, $U_i(X_i, \sigma_{-i}) = \prod_{j \in N} \sigma_j(Y_j) > U_i(Y_i, \sigma_{-i}) = \prod_{j \in N} \sigma_j(X_j)$ is equivalent to $1 > \sum_{j \in N} \sigma_j(X_j)$ where $i \neq j$ and $\sigma_{-i} \in \times_{j \in N} \Delta(A_j)$. Moreover, a player is indifferent between alternative X and alternative Y whenever her opponents' probabilities for one of the two alternatives sum to 1. This justifies the (continuum of) equilibria where $\sigma_i(X_i) \in [0, 1]$ and $\sigma_j(X_j) = 1 - \sigma_k(X_k) = 1$ with $i, j, k \in N$ and $i \neq j \neq k$. In such a case, player $i \in N$ can induce either one of the two pure strategy equilibria. It also justifies the completely mixed equilibrium with $\sigma_i(X_i) = 1/2 \forall i \in N$ which is the only symmetry invariant equilibrium of the minority of three-game. To summarize, the Nash equilibria of the minority of three-game are $(1/2, 1/2, 1/2)$ and all permutations of $(\sigma(X), 1, 0)$ where $\sigma(X) \in [0, 1]$.

Interestingly enough, the standard game-theoretical predictions remain unchanged if one considers a slightly modified version of the minority of three-game where player i 's utility function is given by

$$u_i(a_i, a_{-i}) = \begin{cases} 1 & \text{if } a_i \neq a_j \text{ and } a_i \neq a_k \\ 0 & \text{if } a_i = a_j = a_k \\ x & \text{otherwise,} \end{cases}$$

with $0 \leq x < 1$.³ Indeed, the two games are identical when purely focusing on best-reply behavior. Thus, as long as we rely on solution ideas which only depends on the best-reply structure of the game,

²The minority of three-game belongs to the class of win-loss games. For the sake of completeness, one can denote by π the monetary payoff associated with the less rewarding alternative, i.e., the one chosen by two or more players, and by $\Pi > \pi$ the monetary payoff associated with the most rewarding alternative, i.e., the one chosen by a single player. Obviously, because each player can receive one of only two possible payoffs, there is no opportunity for choices by expected-utility maximizers to be influenced by nonlinearities (risk preferences) in their utility functions. In particular, when players play mixed strategies, all of the induced lotteries are binary lotteries.

³This payoff structure is the one underlying a three players market entry game with two markets, each market having a unitary capacity, and a payoff function decreasing in a linear way.

both payoff structures lead to the same predictions. From now on, we focus on the payoff structure described in Table 1.

2.2 Equilibrium selection

In view of the theories of equilibrium selection, the minority of three-game is quite pathologic since it is one of the rare applications where one encounters a minimal formation containing multiple equilibria (see Harsanyi and Selten, 1988 and Güth and Kalkofen, 1989 for other applications of equilibrium selection theory).

Lemma 1 *The minority of three-game M3G has no proper subformation and is therefore a minimal formation.*

We now demonstrate that this pathology is fundamental in the sense that (i) it is noise persistent, and (ii) asymmetries will not question its solution. The idea of “noise” is to solve the unperturbed game as an idealisation. Asymmetry of “noise” appears to be realistic but is rather arbitrary from a normative perspective.

If we neglect that pure strategy equilibria are represented by equilibria in extreme mixed strategies (all freely disposable probability is put on one choice) the multiplicity of all equilibria applies also to the ε -uniformly perturbed minority of three-game with the restrictions $\sigma_i(X_i) \in [\varepsilon, 1 - \varepsilon]$ for $i = 1, 2, 3$ where $\varepsilon \in (0, 1/2)$ is supposed to be small.

Lemma 2 *For all $\varepsilon \in (0, 1/2)$, the ε -uniformly perturbed minority of three-game has no proper subformation and is therefore a minimal formation.*

Instead of assuming uniform trembles, one can consider asymmetric trembles. More precisely, let us introduce a minor asymmetry in the sense that no two players have the same minimal choice probabilities, which we assume to be the same for both their pure strategies. Let $\varepsilon \in (0, 1/6)$ and assume for the sake of specificity $\sigma_1(X_1) \in [\varepsilon, 1 - \varepsilon]$, $\sigma_2(X_2) \in [2\varepsilon, 1 - 2\varepsilon]$, and $\sigma_3(X_3) \in [3\varepsilon, 1 - 3\varepsilon]$. We refer to the extreme mixtures by σ_i^ε , which are X_1^ε and Y_1^ε for player 1, $X_2^{2\varepsilon}$ and $Y_2^{2\varepsilon}$ for player 2, and $X_3^{3\varepsilon}$ and $Y_3^{3\varepsilon}$ for player 3. We can establish the following result

Lemma 3 *The minority of three-game with the asymmetric trembles ε for player 1, 2ε for player 2 and 3ε for player 3 where $\varepsilon \in (0, 1/6)$ has only two “pure strategy equilibria” (in the sense of using one choice with maximal probability), namely $(Y_1^\varepsilon, X_2^{2\varepsilon}, X_3^{3\varepsilon})$ and $(X_1^\varepsilon, Y_2^{2\varepsilon}, Y_3^{3\varepsilon})$.*

When trying to select a unique solution for the (ε -uniformly perturbed) minority of three-game, the fact that this game has no proper subformation becomes crucial. According to the theory of Harsanyi and Selten (1988), one therefore has to apply the tracing procedure directly to the (ε -uniformly perturbed) minority of three-game to select a unique equilibrium. In view of the complete symmetry of the (ε -uniformly perturbed) minority of three-game as well as of the tracing procedure, this has to select the completely mixed equilibrium according to which all three players use both choices with equal probability. Thus, to avoid the only symmetry invariant solution of the unperturbed game it does not suffice to assume different trembles for different players.⁴

In the asymmetric (trembles) case, the two “pure strategy equilibria” qualify as primitive formations since both equilibria are strict. Since neither of these two solution candidates can (payoff or) risk dominate the other, the theory of Harsanyi and Selten (1988) suggests to neglect them what essentially means to apply the tracing procedure to the full minority of three-games with asymmetric trembles. The degenerate nature of these games will imply that the linear tracing procedure yields no unique result so that one has to apply its logarithmic version what hardly ever is needed in (economic) applications. The symmetry of the two “pure strategy” equilibria (in the sense of using one choice with maximal probability) as well as of the logarithmic tracing procedure implies that the solution for $\varepsilon \rightarrow 0$ prescribes

⁴One might assume different trembles also for different choices what however appears even more arbitrary.

that all three players will use both choices (X and Y) with equal probability. (It is obvious that the sum of expected payoffs over all three players, implied by the solution, is less than that of any of its pure strategy equilibria.⁵)

2.3 Alternative predictions

Below, we establish that the completely mixed equilibrium is also predicted by alternative solution concepts which, arguably, rely on less stringent assumptions regarding the knowledge and understanding of players.

Sampling equilibrium

Contrary to the common approach which is based on the dynamics of evolution and learning, Osborne and Rubinstein (1998) have recently developed a static and equilibrium-based approach to the modeling of bounded rationality in games. Needless to say, this approach relies on less stringent assumptions regarding the knowledge and understanding of players than does the standard theory of Nash equilibrium.⁶ Indeed, each player, rather than optimizing given a belief about the other players' behavior, first associates one consequence with each of her actions by sampling each of her actions K times, $K \in \mathbb{N}^*$, and then she chooses the action that has the best consequence. In a symmetric game, a *sampling*(K) equilibrium ($S(K)$ -equilibrium) is a mixed strategy such that if all other players adopt this strategy throughout the sampling procedure, then the probability that a given action is best under the sampling procedure is precisely the probability with which it is chosen. One interpretation of sampling-equilibria that is advanced by Osborne and Rubinstein is that it is the steady state of a dynamic process involving a large population of individuals who are randomly matched to play the game. Each member of the population adopts the same action throughout her stay in the population, and the population composition changes as a result of new entrants and departures. When entering, a player samples each action K times and selects that which yields the best outcome according to the procedure described above. In this case, an $S(K)$ -equilibrium is a distribution of actions in the incumbent population which induces the same distribution of actions in the flow of entrants.⁷

In the minority of three-game, if player $i \in N$ samples both available actions (X_i and Y_i) K times, the probability that action X_i yields the best outcome is given by

$$\sum_{k_1=1}^K \text{Prob}[u(X_i, \sigma_{-i}) = k_1] \left(\sum_{k_2=0}^{k_1-1} \text{Prob}[u(Y_i, \sigma_{-i}) = k_2] \right) + \frac{1}{2} \sum_{k=0}^K \text{Prob}[u(X_i, \sigma_{-i}) = k] \text{Prob}[u(Y_i, \sigma_{-i}) = k],$$

where in the case of realizations in which X_i is not unique in yielding the best outcome, the probability is weighted by 1/2. This winning probability can be rewritten

$$\begin{aligned} & \sum_{k_1=1}^K \binom{K}{k_1} (\sigma_j(Y_j) \sigma_k(Y_k))^{k_1} (1 - \sigma_j(Y_j) \sigma_k(Y_k))^{K-k_1} \left(\sum_{k_2=0}^{k_1-1} \binom{K}{k_2} (\sigma_j(X_j) \sigma_k(X_k))^{k_2} (1 - \sigma_j(X_j) \sigma_k(X_k))^{K-k_2} \right) \\ & + \frac{1}{2} \sum_{k=0}^K \binom{K}{k} (\sigma_j(Y_j) \sigma_k(Y_k))^k (1 - \sigma_j(Y_j) \sigma_k(Y_k))^{K-k} \binom{K}{k} (\sigma_j(X_j) \sigma_k(X_k))^k (1 - \sigma_j(X_j) \sigma_k(X_k))^{K-k}. \end{aligned}$$

As already mentioned, a sampling-equilibrium of a symmetric game corresponds to the steady state of a dynamic process involving a large *single* population of individuals who are randomly matched to play the game. An $S(K)$ -equilibrium of the minority of three-game is a probability distribution

⁵Each player's expected payoff at the symmetric equilibrium equals 1/4 which leads to an expected total payoff of 3/4 for the three players.

⁶Actually, each player must know only her own set of actions.

⁷Sethi (2000) formalizes this dynamic process and uses the criterion of dynamic stability as an equilibrium refinement.

$(\sigma(X), \sigma(X), \sigma(X)) \in [0, 1]^3$ such that

$$\begin{aligned} \sigma(X) = & \sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} \left(\sum_{k_2=0}^{k_1-1} \binom{K}{k_2} \sigma(X)^{2k_2} (1 - \sigma(X)^2)^{K-k_2} \right) \\ & + \frac{1}{2} \sum_{k=0}^K \binom{K}{k} (1 - \sigma(X))^{2k} (\sigma(X) (2 - \sigma(X)))^{K-k} \binom{K}{k} \sigma(X)^{2k} (1 - \sigma(X)^2)^{K-k}. \end{aligned}$$

According to Osborne and Rubinstein's (1998) corollary [page 844], the equilibrium mixed strategy of the minority of three-game is the unique limit of $S(K)$ -equilibria as $K \rightarrow \infty$. In fact, this equivalence holds for every level of sampling, not only in the limit.

Lemma 4 *For each $K \in \mathbb{N}^*$, the unique $S(K)$ -equilibrium of the minority of three-game is given by $(1/2, 1/2, 1/2)$.*

Impulse balance equilibrium

Impulse balance equilibrium is based on a simple principle of ex-post rationality. It applies to all games in which players repeatedly decide on one parameter and in which the feedback environment allows conclusions about what would have been the better choice in the last interaction.⁸ In the minority of three-game, the parameter is the probability of choosing one of the two alternatives, which can be adjusted upward or downward. Expected upward and downward impulses are equal for each of the three players simultaneously in impulse balance equilibrium.

Lemma 5 *The set of impulse balance equilibria is identical to the set of Nash equilibria of the minority of three-game: it consists of $(1/2, 1/2, 1/2)$ and all permutations of $(\sigma(X), 1, 0)$ where $\sigma(X) \in [0, 1]$.*

3 Experimental design

The experiment consists of four sessions, with 27 subjects in each session, for a total of 108 subjects. Subjects played 50 repetitions the minority of three-game, they were randomly rematched after each play, and earnings, derived from the payoff numbers in the previous section, were recorded in points (the experimental currency).

In each repetition of the game, subjects were asked to give a probability distribution over the two alternatives (X and Y) instead of picking an alternative. A single random draw was made from this distribution, and the realization became the subject's alternative. This mixed strategy device allows subjects to generate random play through a probability experiment that they control and conduct on the computer. Each subject had the option to fill an urn of 100 balls with any composition of alternatives (balls) he or she desires. Once the urn was filled, the computer randomly selected one of the 100 balls as the chosen alternative. However, opponents were only shown the chosen alternative (ball), not the mixed strategy (i.e., the composition of the urn). This generation of a random outcome is in the spirit of how mixed strategies are motivated in the classical treatments of game theory; namely, players choose a probability distribution over the set of alternatives, and then draw a realization. The mixed strategy device provides benefits to both the experimenter and the subjects. First, this device allows subjects to easily generate i.i.d. sequences of alternatives across stage games, or in other words, successfully execute intended mixed strategies. Second, the device also provides the researcher with a new view of how subjects may actually be playing the game.

In each repetition of the game, subjects had the possibility to collect some information about the five previous repetitions. Concretely, subjects had access to: (i) their choice and their earnings; (ii)

⁸Ockenfels and Selten (2005) show that the impulse equilibrium concept captures the experimental data of sealed-bid first-price auctions with private values.

the percentage of their interacting opponents who chose alternative ‘X’ and the percentage of their interacting opponents who chose alternative ‘Y’; (iii) the average earnings of their interacting opponents who chose ‘X’ and the average earnings of their interacting opponents who chose ‘Y’. These information-gathering data illuminate the behavioral rules subjects are using and enable an indirect test of whether they are best-replying (belief-based learning).

Practical procedures

The four sessions of the computerized experiment were conducted at the Experimental Laboratory of the Max Planck Institute of Economics in Jena (Germany). The experiment was programmed and conducted with the software z-Tree (Fischbacher, 2007). Subjects were invited using an Online Recruitment System (Greiner, 2004). All 108 subjects were undergraduate students from various disciplines at the University of Jena, in each session the gender composition was approximately balanced, and no subject participated in more than one session. Some subjects had participated in earlier economics experiments, but all were inexperienced in the sense that they had never taken part in an earlier session of this type. Each session lasted on average slightly less than 2 hours and the average earnings per subject were about 15 euros (about \$22), including a 2.50 euros show-up fee.⁹

At the beginning of each session subjects randomly drew a cubicle number. Once all subjects sat down in their cubicles, instructions were distributed. Cubicles were visually isolated from each other and communication between the subjects was strictly prohibited. Subjects first read the instructions silently and then listened as the monitor read them aloud (the monitor was a native German speaker). Questions were answered privately. A short control questionnaire and two training repetitions followed.¹⁰ After all subjects had answered correctly the control questionnaire, subjects played 50 repetitions of the minority of three-game. Subjects were told that they would interact with randomly changing opponents. Actually in each session there were three independent matching groups with nine participants. Subjects played against randomly chosen opponents but only within their independent group. They were not informed about the fact that there are three groups. We did not lie to them but conveyed the impression that they interact with 26 other players. After each repetition of the minority of three-game, the computer screen displayed the alternative chosen by each of the three players as well as the three earnings. Subjects were not permitted to take notes of any kind about their playing experience. At the end of the 50 repetitions, subjects’ payoffs were displayed on their screens and subjects privately retrieved their final earnings (including the show-up fee).

4 Results

In this section, we attempt at characterizing the decision rules used by participants. First, we provide some aggregate statistics of our experimental data. At the aggregate level, players’ behavior seems based on the toss of a fair coin. Second, we investigate individual behavior. To do so, we introduce a series of learning models and test the predictions of these models against our experimental data at the individual level. Our individual-data analysis strongly indicates the existence of heterogeneous decision rules among participants.

4.1 Aggregate statistics

Table 2 reports the matching group-level means and standard deviations in per repetition payoffs and chosen number of X -balls. In a given repetition of the game, the number of X -balls chosen by the

⁹Points were converted to euros in the calculation of subjects’ final earnings at a conversion rate of 1 point to 1 euro.

¹⁰We took subjects through two training repetitions to familiarize them with the software, especially the mixed strategy device. During the two trial repetitions, subjects were not able to choose freely the composition of the urn. Indeed, in trial repetition 1 the urn had to consist of 99 ‘X’ balls and 1 ‘Y’ ball whereas in trial repetition 2 the urn had to consist of 1 ‘X’ ball and 99 ‘Y’ balls. Subjects whose questionnaire results indicated that they had not sufficiently understood the rules of the game were replaced and paid 5 euros for answering the questionnaire (35 subjects were invited for each session).

subject might be interpreted as her/his mixed strategy. Clearly, observed behavior is very much in line with the predictions of the completely mixed equilibrium at the *aggregate level*.

Matching group	Payoff		Number of X-balls	
	Mean	Std. Dev.	Mean	Std. Dev.
1	0.258	0.438	49.042	31.081
2	0.256	0.437	49.769	33.050
3	0.247	0.432	52.900	37.387
4	0.229	0.421	48.049	32.430
5	0.242	0.429	47.958	40.439
6	0.253	0.435	49.573	36.756
7	0.269	0.444	49.613	41.347
8	0.251	0.434	48.776	36.404
9	0.264	0.441	53.271	38.950
10	0.255	0.437	44.042	37.381
11	0.260	0.439	47.380	42.629
12	0.260	0.439	50.198	33.481
Mean	0.254	0.435	49.214	36.778

Table 2: Payoffs and mixed strategies at the matching group level.

Figure 1 shows the distribution of the changes of X -balls in successive repetitions. Five peaks are worth noticing. The by far highest peak at 0 reveals that in more than 50 percent of the cases there is no change towards the next repetition. The other peaks at -100 and 100 reveal that switches between the pure strategies occur in slightly more than 10 percent of the cases. The peaks at -50 and 50 reveal switches from the symmetric equilibrium to one of the pure strategies or vice-versa. Those switches occur in less than 10 percent of the cases.

Finally, Figure 2 illustrates the temporal paths of X -balls choices for three different participants in the first matching group. Participant 4 chose in every repetition 50 X -balls whereas participant 9 alternated his/her choice of X -balls between 100 and 0. Participant 6 is an example of a player who avoided the extrema. This illustration suggests the existence of heterogeneous decision rules which is confirmed in our individual-data analysis below.

4.2 The learning models

Apart from the two varieties of learning which received the most scrutiny in experiments, belief learning models and reinforcement learning models, we also consider learning models which formalize the dynamic processes that might lead to the alternative equilibria of Section 2.

Belief-based learning

One widely used model of learning is the process of fictitious play (FP).¹¹ In this process, players behave as if they think they are facing a stationary, but unknown, distribution of opponents strategies. Initially, each opponent's alternative is equally likely to be chosen. In each repetition, players choose a pure strategy that is a best response to the belief formed from weighted average of immediate past and the history before it, and they randomize when indifferent. In repetition $t + 1$, the weight associated with the immediate past equals $1/t$.

¹¹See Fudenberg and Levine (1998) for details.

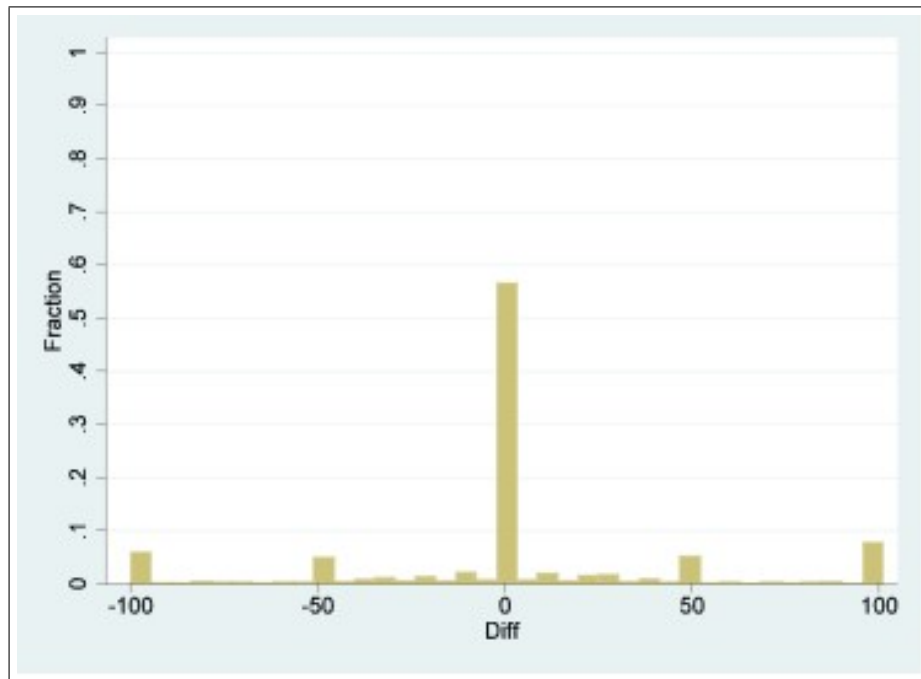


Figure 1: Distribution of the changes in X -balls choices in two successive repetitions.

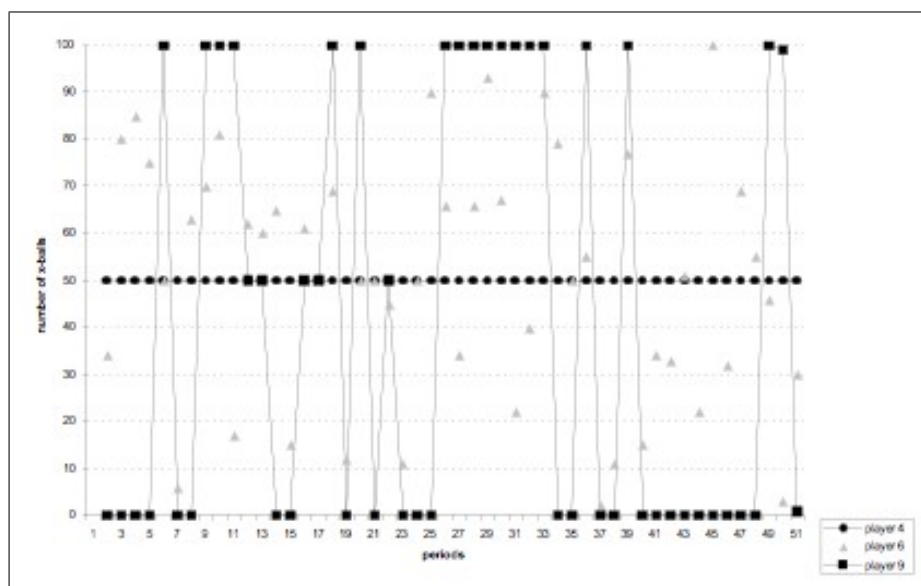


Figure 2: Temporal paths of X -balls choices.

As a special case, we also consider the Cournot adjustment model (Cournot) where players choose a pure strategy that is a best response to the belief formed from immediate past.

Reinforcement learning

An alternative, very elementary type of learning, is reinforcement learning (RL) which became recently the subject of ongoing experimental research in economics (see, e.g., Erev and Roth, 1998). Players associate with each alternative a propensity and the two propensities are set equal to one in the first repetition. After each repetition, actual payoffs are added to propensities (clearly, for a given player, nothing is added to the propensity which corresponds to the alternative not chosen). Players choose an alternative according to the mixed strategy given by the ratios of the two propensities to their sum.

Self-tuning experience weighted attraction learning

Camerer and Ho's (1999) experience-weighted attraction learning model is a hybrid model that encompasses several belief-based and reinforcement learning models as special cases. Unfortunately, this learning model has many parameters which makes it difficult to compare with other (simpler) learning models. We consider the one-parameter version of this learning model, named the self-tuning experience weighted attraction learning model (STEWAL), and we refer the reader to Ho, Camerer, and Chong (2007) for full details.

The response sensitivity is a parameter which needs to be calibrated before predictions can be compared to individual behavior. We allow the response sensitivity to take any value in the interval $]0, \infty[$.

Impulse balance learning

Impulse balance learning (IBL) relates to the concepts of impulse balance equilibrium and learning direction theory (Selten, 1998). Like in reinforcement learning, players associate with each alternative a propensity (initially, the two propensities are set equal to one) and alternatives are chosen according to the mixed strategy given by the ratios of the two propensities to their sum. However, propensities are updated differently than in reinforcement learning. In particular, only the propensity of the non-chosen alternative might be updated. Suppose that the alternative chosen by the player in a given repetition is not the best reply to the pair of alternatives chosen by the opponents. Then the difference between the best-reply payoff and the actual payoff is added to the propensity of the non-chosen alternative. There is no updating of propensities whenever the alternative chosen is the best reply.

Impulse matching learning

Impulse matching learning (IML) is identical to impulse balance learning except that the difference between the best-reply payoff and the actual payoff is *always* added to the propensity of the ex-post optimal alternative.

Payoff-sampling learning

Payoff-sampling learning (PSL) relates to the concept of sampling equilibrium exposed in Section 2. In each repetition, players first draw one sample of earlier payoffs for each alternative where the samples are randomly drawn with replacement. Second, the cumulated payoffs of each sample are computed and the alternative with largest payoffs is chosen (if cumulated payoffs are identical, players randomize). Initially and until positive payoffs for each alternative have been obtained at least once, players randomize.

The sample size is a parameter which needs to be calibrated before predictions can be compared to individual behavior. We allow the sample size to take any value between 2 and 7.

4.3 Individual decision rules

To assess the descriptive power of a given learning model LM , we compute for each subject i and in each repetition $t \in \{1, \dots, 50\}$ the mean-squared deviation

$$D_i^{LM}(t) = \left(p_X^{LM}(t) - \frac{X\text{-balls}_i(t)}{100} \right)^2 \quad (1)$$

between the predicted probability $p_X^{LM}(t)$ of the learning model and the ratio $X\text{-balls}_i(t)/100$ chosen by the subject in the respective repetition. The mean deviation score (MDS) is given by

$$MDS_i^{LM} = \sqrt{\frac{\sum_t D_i^{LM}(t)}{50}} \quad (2)$$

and is a measure of the goodness of fit for learning model LM and subject i . Mean deviation scores were computed for each of the considered learning models as well as for the symmetric equilibrium strategy (SES) where the player chooses each alternative with equal probability ($X\text{-balls} = 50 \forall t$). In the following, we compare the MDS with the early MDS (repetitions 1 to 17), the middle MDS (repetitions 18 to 34), and the late MDS (repetitions 35 to 50).

Figure 3 reports the mean deviations scores averaged over all 108 participants. For most learning models as well as for the symmetric equilibrium strategy, minor differences are observed between the predictive power of early, middle and late play. Still, in the case of reinforcement learning, self-tuning experience weighted attraction learning, impulse matching learning and the symmetric equilibrium strategy, the predictive power clearly increases during the course of the session. We performed pairwise comparisons of these scores with the help of two-tailed matched-pairs Wilcoxon signed rank tests. These statistical comparisons, summarized in Table 3, confirm the existence of heterogeneous decision rules among our participants. The three best learning models are self-tuning experience weighted attraction learning, impulse matching learning and reinforcement learning and their predictive power is comparable to the one achieved by the symmetric equilibrium strategy. Note that the mean deviation score of self-tuning experience weighted attraction learning is minimized when the response sensitivity equals 0.32 whereas the mean deviation score of payoff-sample learning is minimized when the sample size equals 2.

	RL	STEWAL	SES	IML	IBL	PSL	FP
STEWAL	n.s.						
SES	n.s.	n.s.					
IML	n.s.	n.s.	n.s.				
IBL	< 0.05	< 0.10	< 0.10	n.s.			
PSL	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01		
FP	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	
Cournot	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	< 0.01	n.s.

Table 3: P -values in favor of column models, two-tailed matched-pairs Wilcoxon signed rank test.

Next we classify participants' behavior over the 50 repetitions of a session. Reinforcement learning is the most successful decision rule as it describes best the behavior of 29% of our participants. Almost as successful are self-tuning experience weighted attraction learning and the symmetric equilibrium strategy which describe best the behavior of 24% of our participants. Impulse matching learning describes best the behavior of only 16% of our participants and none of the remaining decision rules describes best the behavior of more than 5% of our participants.

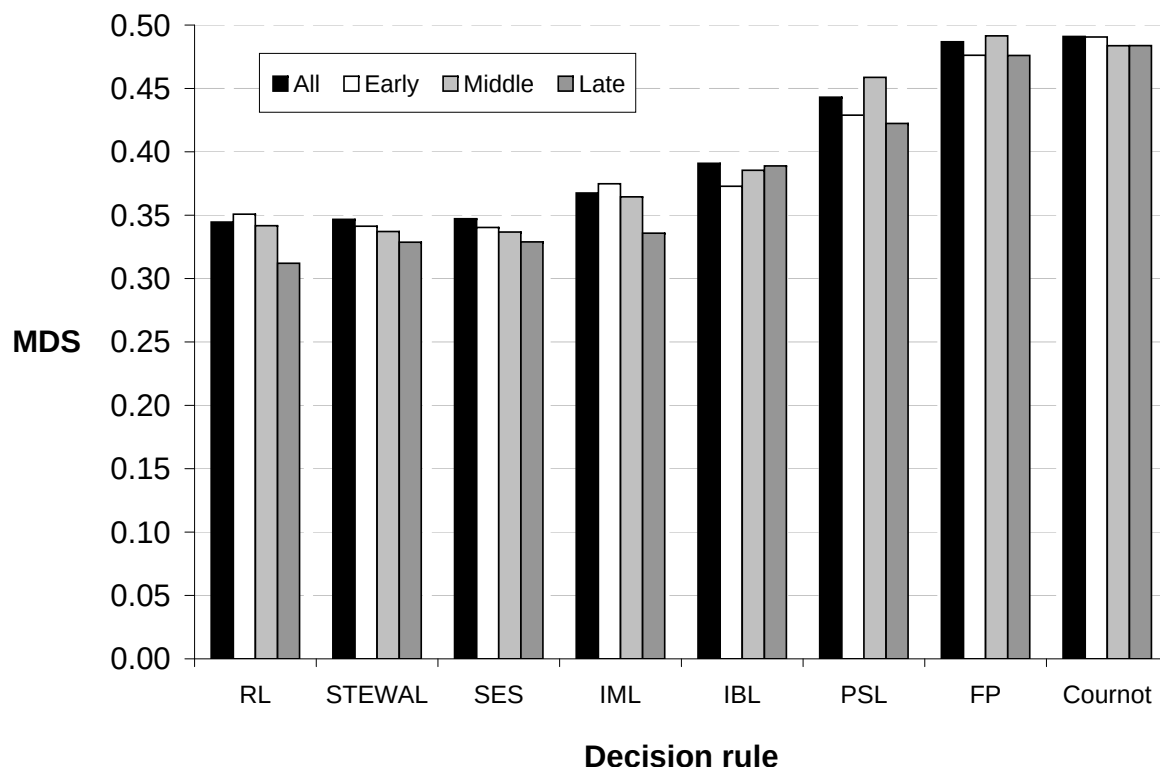


Figure 3: Mean deviation scores averaged over all 108 participants.

Our analyzes of the goodness of fit indicate the existence of heterogeneous decision rules and the low predictive power of belief learning models. The latter result is confirmed by the analysis of participants' information retrieval. Participants mainly retrieved information concerning their own payoff as well as their opponents' payoffs in the previous repetition. Few information retrievals were made about earlier repetitions than the previous one.

5 Related Literature

Though the minority game is related to market entry games, its equilibrium structure differs and favors the completely mixed equilibrium as a natural benchmark. Thus, the numerous evidence gathered by experimental economists on behavior in market entry games (e.g. Duffy and Hopkins, 2005) might not be transferable to minority games. Another related game is the route-choice game experimentally studied by Selten, Chmura, Pitz, Kube, and Schreckenberg (2007). Again, the likelihood of observing convergence to a pure strategy equilibrium is larger than in the minority game since pure strategy equilibria induce the same payoff for all players.

Few experimental studies have been conducted by economists on the minority game. Chmura and Pitz (2006) report on two experimental treatments that differ in the amount of information given to participants. The more information participants get, the more often they stick to their choice in the next repetition. Chmura and Pitz (2010) compares the behavior of Chinese and German participants and show that Chinese participants exhibit a stronger tendency to stick to their choice. Bottazzi and Devetag (2007) investigates the extent to which *stationary* groups of five participants are able to coordinate efficiently in a repeated minority game, and the impact of information on the resulting efficiency. Groups achieve a payoff level equal or higher than the one associated with the completely mixed equilibrium, and participants use public information as a coordination device. At the individual level, little evidence of behavioral consistency with the completely mixed equilibrium is found, but strong evidence of dynamic adaptation. Our experimental study confirms the finding that there is considerable heterogeneity in participants' behavior. However, by implementing a *strangers design* and

endowing participants with a mixing device, we give the predictions of the completely mixed equilibrium their “best shot”. Our results are damaging evidence against the behavioral relevance of the symmetric equilibrium strategy since participants’ choices do not result from their inability to generate random sequences of actions, and they are unlikely to be the result of repeated game strategies.

6 Conclusion

Market entry games are prototypical of coordination problems arising from a newly emergent profit opportunity that can be exploited only by a limited number of individuals. Many experimental studies have been conducted in an effort to find out which type of equilibrium participants are likely to coordinate upon. However, none of these experimental studies has yielded evidence to suggest that participants play equilibrium strategies. We pursue this line of research by conducting an experiment on the minority of three-game. The latter game has many asymmetric equilibria in pure strategies that are non strict and payoff-asymmetric but a unique symmetric mixed strategy equilibrium, in which each player selects the two actions with equal probability. We show that such a straightforward behavior is predicted by Harsanyi and Selten’s (1988) equilibrium selection theory as well as alternative solution concepts like impulse balance equilibrium and sampling equilibrium. The completely mixed equilibrium seems therefore to be the natural benchmark to which we can compare participants’ behavior. We give the predictions of the completely mixed equilibrium their “best shot” by implementing a strangers design and endowing participants with a mixing device, and we also offer participants the possibility to collect some information about previous play.

Our results indicate that participants rely on various decision rules, and that only a quarter of them decide according to the toss of a fair coin. Reinforcement learning is the most successful decision rule as it describes best the behavior of about a third of our participants. Belief learning models have low predictive success which is line with the fact that participants mainly collect information about past payoffs.

In conclusion, heterogeneity in behavioral rules seems a persistent fact of games with many equilibrium outcomes, and this considerable heterogeneity in behavior does not result only from participants’ inability to generate random sequences of actions or from repeated game considerations.

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Appendix: Proofs

Proof of Lemma 1. A **formation** of $M3G$ is a substructure $F = (F_1, F_2, F_3, u_1(\cdot)|_F, u_2(\cdot)|_F, u_3(\cdot)|_F)$ with $\emptyset \neq F_i \subseteq A_i$ for $i = 1, 2, 3$ and $u_i(\cdot)|_F$ denoting the restriction of $u_i(\cdot)$ to strategy vectors $a \in \times_{j \in N} F_j$, which is closed with respect to best replies. Such a formation F of $M3G$ is minimal when there exists no proper subformation F' of F .

If F is a proper substructure of $M3G$ then there exists a player $i \in \{1, 2, 3\}$ with either $F_i = \{X_i\}$ or $F_i = \{Y_i\}$. Without loss of generality, we assume that $F_i = \{X_i\}$. To show that such a substructure is no formation, we simply distinguish the possible cases where we denote by j and k the two opponents of player i .

- If $|F_j| = |F_k| = 1$, i.e., all three players have only one strategy, then the two possibilities are:
 1. F_j and F_k contain the same choice (X or Y): If F_i also contains the same choice then player i 's best reply is not contained in F_i . If F_i contains a different choice then both pure strategies are best replies for player j and player k . In both cases, F does not qualify as a formation.
 2. F_j and F_k contain different choices: Without loss of generality, we assume that F_i and F_j contain the same choice. Clearly, both pure strategies are best replies for player i and player j . F does not qualify as a formation.
- At least one of the two sets F_j or F_k contains two strategies where, without loss of generality, we assume that this is F_j .
 1. If F_i and F_k contain the same choice, player i will want to use his strategy, not contained in F_i , when j uses the same choice as k . Thus, F is no formation.
 2. If F_i and F_k contain different choices, for both of them their in F non-feasible action is a best reply if j chooses the same alternative as the other (the choice in F_i , respectively F_k , when considering player i , respectively k). Again, F cannot be closed with respect to best replies.

□

Proof of Lemma 2. For a given $\varepsilon \in (0, 1/2)$, we denote by a_i^ε , $a_i \in A_i = \{X_i, Y_i\}$, the extreme mixed strategy with $\sigma_i(a_i) = 1 - \varepsilon$ ($\sigma_i(b_i) = \varepsilon$ where $b_i \in A_i = \{X_i, Y_i\}$ and $b_i \neq a_i$). The normal-form representation of the ε -uniformly perturbed minority of three-game is given by

X_3^ε

		Player 2	
		X_2^ε	Y_2^ε
Player	X_1^ε	α, α, α	α, β, α
1	Y_1^ε	β, α, α	α, α, β

Player 3

Y_3^ε

		Player 2	
		X_2^ε	Y_2^ε
Player	X_1^ε	α, α, β	β, α, α
1	Y_1^ε	α, β, α	α, α, α

Table 4: The ε -uniformly perturbed minority of three-game in normal form.

where $\alpha = \varepsilon(1 - \varepsilon)$ and $\beta = (1 - \varepsilon)^3 + \varepsilon^3$. As $(1 - \varepsilon)^3 + \varepsilon^3 - \varepsilon(1 - \varepsilon) = (1 - 2\varepsilon)^2 > 0$, we can transform this bimatrix by the positively affine utility transformation $\tilde{u}_i(\cdot) = (u_i(\cdot) - \varepsilon(1 - \varepsilon))/((1 - 2\varepsilon)^2)$ for $i = 1, 2, 3$ to obtain the same bimatrix representation as for the non perturbed minority of three-game.

□

Proof of Lemma 3. The payoff structure of the minority of three-game with the considered asymmetric trembles is given by

- $u_1(X_1^\varepsilon, X_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_1(Y_1^\varepsilon, Y_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon(6\varepsilon(1-\varepsilon) + (1-2\varepsilon)(1-3\varepsilon));$
- $u_2(X_1^\varepsilon, X_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_2(Y_1^\varepsilon, Y_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon(3\varepsilon(1-2\varepsilon) + 2(1-\varepsilon)(1-3\varepsilon));$
- $u_3(X_1^\varepsilon, X_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_3(Y_1^\varepsilon, Y_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon(2\varepsilon(1-3\varepsilon) + 3(1-\varepsilon)(1-2\varepsilon));$
- $u_1(X_1^\varepsilon, Y_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_1(Y_1^\varepsilon, X_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon(3(1-\varepsilon)(1-2\varepsilon) + 2\varepsilon(1-3\varepsilon));$
- $u_2(X_1^\varepsilon, Y_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_2(Y_1^\varepsilon, X_2^{2\varepsilon}, Y_3^{3\varepsilon}) = (1-\varepsilon)(1-2\varepsilon)(1-3\varepsilon) + 6\varepsilon^3;$
- $u_3(X_1^\varepsilon, Y_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_3(Y_1^\varepsilon, X_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon((1-2\varepsilon)(1-3\varepsilon) + 6\varepsilon(1-\varepsilon));$
- $u_1(Y_1^\varepsilon, X_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_1(X_1^\varepsilon, Y_2^{2\varepsilon}, Y_3^{3\varepsilon}) = (1-\varepsilon)(1-2\varepsilon)(1-3\varepsilon) + 6\varepsilon^3;$
- $u_2(Y_1^\varepsilon, X_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_2(X_1^\varepsilon, Y_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon(3(1-2\varepsilon)(1-\varepsilon) + 2\varepsilon(1-3\varepsilon));$
- $u_3(Y_1^\varepsilon, X_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_3(X_1^\varepsilon, Y_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon(2(1-3\varepsilon)(1-\varepsilon) + 3\varepsilon(1-2\varepsilon));$
- $u_1(Y_1^\varepsilon, Y_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_1(X_1^\varepsilon, X_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon(2(1-\varepsilon)(1-3\varepsilon) + 3\varepsilon(1-2\varepsilon));$
- $u_2(Y_1^\varepsilon, Y_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_2(X_1^\varepsilon, X_2^{2\varepsilon}, Y_3^{3\varepsilon}) = \varepsilon((1-2\varepsilon)(1-3\varepsilon) + 6\varepsilon(1-\varepsilon));$
- $u_3(Y_1^\varepsilon, Y_2^{2\varepsilon}, X_3^{3\varepsilon}) = u_3(X_1^\varepsilon, X_2^{2\varepsilon}, Y_3^{3\varepsilon}) = (1-\varepsilon)(1-2\varepsilon)(1-3\varepsilon) + 6\varepsilon^3.$

Straightforward computations show that the asymmetrically perturbed minority of three-game exhibits two strict “pure strategy equilibria”: $(Y_1^\varepsilon, X_2^{2\varepsilon}, X_3^{3\varepsilon})$ and $(X_1^\varepsilon, Y_2^{2\varepsilon}, Y_3^{3\varepsilon})$.

□

Proof of Lemma 4. As already mentioned, an $S(K)$ -equilibrium of the minority of three-game is a probability distribution $(\sigma(X), \sigma(X), \sigma(X)) \in [0, 1]^3$ such that

$$\begin{aligned} \sigma(X) &= \sum_{k_1=1}^K \binom{K}{k_1} (1-\sigma(X))^{2k_1} (\sigma(X)(2-\sigma(X)))^{K-k_1} \left(\sum_{k_2=0}^{k_1-1} \binom{K}{k_2} \sigma(X)^{2k_2} (1-\sigma(X)^2)^{K-k_2} \right) \\ &\quad + \frac{1}{2} \sum_{k=0}^K \binom{K}{k} (1-\sigma(X))^{2k} (\sigma(X)(2-\sigma(X)))^{K-k} \binom{K}{k} \sigma(X)^{2k} (1-\sigma(X)^2)^{K-k}. \end{aligned}$$

Below, we rewrite this equality. First,

$$\begin{aligned} \sigma(X) &= \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1-\sigma(X))^{2k_1} (\sigma(X)(2-\sigma(X)))^{K-k_1} \left(\sum_{k_2=0}^{k_1-1} \binom{K}{k_2} \sigma(X)^{2k_2} (1-\sigma(X)^2)^{K-k_2} \right) \\ &\quad + \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1-\sigma(X))^{2k_1} (\sigma(X)(2-\sigma(X)))^{K-k_1} \left(\sum_{k_2=0}^{k_1-1} \binom{K}{k_2} \sigma(X)^{2k_2} (1-\sigma(X)^2)^{K-k_2} \right) \\ &\quad + \frac{1}{2} \sum_{k=0}^K \binom{K}{k} (1-\sigma(X))^{2k} (\sigma(X)(2-\sigma(X)))^{K-k} \binom{K}{k} \sigma(X)^{2k} (1-\sigma(X)^2)^{K-k}. \end{aligned}$$

Second, $\sigma(X)^2 + (1 - \sigma(X)^2) = 1$ implies that

$$\begin{aligned} \sum_{k_2=0}^{k_1-1} \binom{K}{k_2} (\sigma(X)^2)^{k_2} (1 - \sigma(X)^2)^{K-k_2} &= 1 - \binom{K}{k_1} (\sigma(X)^2)^{k_1} (1 - \sigma(X)^2)^{K-k_1} \\ &\quad - \sum_{k_2=k_1+1}^K \binom{K}{k_2} (\sigma(X)^2)^{k_2} (1 - \sigma(X)^2)^{K-k_2} \end{aligned}$$

so that the equality can be rewritten as

$$\begin{aligned} \sigma(X) &= \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} \\ &\quad - \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} \binom{K}{k_1} \sigma(X)^{2k_1} (1 - \sigma(X)^2)^{K-k_1} \\ &\quad - \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} \left(\sum_{k_2=k_1+1}^K \binom{K}{k_2} \sigma(X)^{2k_2} (1 - \sigma(X)^2)^{K-k_2} \right) \\ &\quad + \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} \left(\sum_{k_2=0}^{k_1-1} \binom{K}{k_2} \sigma(X)^{2k_2} (1 - \sigma(X)^2)^{K-k_2} \right) \\ &\quad + \frac{1}{2} \sum_{k=0}^K \binom{K}{k} (1 - \sigma(X))^{2k} (\sigma(X) (2 - \sigma(X)))^{K-k} \binom{K}{k} \sigma(X)^{2k} (1 - \sigma(X)^2)^{K-k}. \end{aligned}$$

Finally, $(1 - \sigma(X))^2 + (\sigma(X) (2 - \sigma(X))) = 1$ implies that

$$\sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} = 1 - \sigma(X) (2 - \sigma(X))^K$$

so that the equality can be rewritten as

$$\begin{aligned} \sigma(X) &= \frac{1}{2} - \frac{1}{2} (\sigma(X) (2 - \sigma(X)))^K + \frac{1}{2} (\sigma(X) (2 - \sigma(X)))^K (1 - \sigma(X)^2)^K \\ &\quad + \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} (g(\sigma(X), K) - h(\sigma(X), K)), \end{aligned}$$

with $g(\sigma(X), K) = \sum_{k_2=0}^{k_1-1} \binom{K}{k_2} \sigma(X)^{2k_2} (1 - \sigma(X)^2)^{K-k_2}$

and $h(\sigma(X), K) = \sum_{k_2=k_1+1}^K \binom{K}{k_2} \sigma(X)^{2k_2} (1 - \sigma(X)^2)^{K-k_2}$. A final simplification leads to

$$\begin{aligned} \sigma(X) &= \frac{1}{2} - \frac{1}{2} (\sigma(X) (2 - \sigma(X)))^K \left(1 - (1 - \sigma(X)^2)^K \right) \\ &\quad + \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} (g(\sigma(X), K) - h(\sigma(X), K)). \end{aligned}$$

Clearly, for a given $K \in \mathbb{N}^*$, $(\sigma(X), \sigma(X), \sigma(X)) \in [0, 1]^3$ is an $S(K)$ -equilibrium of the minority of three-game if and only if $\sigma(X) \in [0, 1]$ is a fixed point of $f(\sigma(X), K)$ where $f(\sigma(X), K)$ is the right-hand side of the above equality.

We now show by induction that for each $K \in \mathbb{N}^*$, $1/2$ is indeed a fixed point of $f(\cdot, K)$. For $K = 1$, one can easily check that $f(1/2, 1) = 1/2$. Additionally, one can show that (details are available from the authors upon request)

$$\begin{aligned} f(1/2, K+1) &= f(1/2, K) - \frac{1}{2} * \frac{1}{4} * \left(\left(\frac{3}{4} \right)^{2K+1} + \left(\frac{1}{4} \right)^K - \left(\frac{1}{4} \right)^{2K} - \left(\frac{1}{4} \right)^K + \left(\frac{1}{4} \right)^{2K+1} \right) \\ &\quad + \frac{1}{2} * \frac{27}{16} \sum_{k_1=1}^K \binom{K}{k_1-1} \binom{K}{k_1-1} \left(\frac{1}{4} \right)^{2k_1} \left(\frac{3}{4} \right)^{2(K-k_1)} \\ &\quad - \frac{1}{2} * \frac{3}{16} \sum_{k_1=1}^K \binom{K}{k_1} \binom{K}{k_1} \left(\frac{1}{4} \right)^{2k_1} \left(\frac{3}{4} \right)^{2(K-k_1)}. \end{aligned}$$

Assuming that, for each $K \in \mathbb{N}^*$, $f(1/2, K) = 1/2$ the above equality simplifies to $f(1/2, K+1) = 1/2$.

Finally, we show that for each $K \in \mathbb{N}^*$, $f(\sigma(X), K)$ decreases on the interval $[0, 1]$. Remember that

$$\begin{aligned} f(\sigma(X), K) &\equiv \frac{1}{2} - \frac{1}{2} (\sigma(X) (2 - \sigma(X)))^K \left(1 - (1 - \sigma(X)^2)^K \right) \\ &\quad + \frac{1}{2} \sum_{k_1=1}^K \binom{K}{k_1} (1 - \sigma(X))^{2k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} (g(\sigma(X), K) - h(\sigma(X), K)). \end{aligned}$$

First, for each $K \in \mathbb{N}^*$, the second term of the function decreases on the interval $[0, 1]$ because: (i) $(\sigma(X) (2 - \sigma(X)))^K$ increases on the interval $[0, 1]$; (ii) $\sigma(X)^2$ clearly increases on the interval $[0, 1]$ which implies that $(1 - \sigma(X)^2)^K$ decreases on the interval $[0, 1]$ and therefore $(1 - (1 - \sigma(X)^2)^K)$ increases on the interval $[0, 1]$.

Second, we show that the third term of the function also decreases on the interval $[0, 1]$. Let $\tilde{b}_1$ be a binomially distributed random variable whose probability distribution is given by $\mathcal{B}(K, \sigma(X)^2)$. $g(\sigma(X), K) = \sum_{k_2=0}^{K-1} \binom{K}{k_2} \sigma(X)^{2k_2} (1 - \sigma(X)^2)^{K-k_2}$ corresponds to $\Pr(\tilde{b}_1 \leq k-1)$ and $h(\sigma(X), K) = \sum_{k_2=k_1+1}^K \binom{K}{k_2} \sigma(X)^{2k_2} (1 - \sigma(X)^2)^{K-k_2}$ corresponds to $P(\tilde{b}_1 \geq k+1)$. When $\sigma(X)$ (and therefore $\sigma(X)^2$) increases, the probability of success increases, so K independent trials lead to more successes. Accordingly, for each $k_1 \in \{1, \dots, K\}$, $g(\sigma(X), K) - h(\sigma(X), K)$ decreases when $\sigma(X)$ increases. Let $\tilde{b}_2$ be a binomially distributed random variable whose probability distribution is given by $\mathcal{B}(K, (1 - \sigma(X))^2)$. $\binom{K}{k_1} \left((1 - \sigma(X))^2 \right)^{k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1}$ corresponds to $\Pr(\tilde{b}_2 = k_1)$. When $\sigma(X)$ increases, $(1 - \sigma(X))^2$ decreases, so $\Pr(\tilde{b}_2 = k_1)$ increases for small values of k_1 and decreases for large values of k_1 . Hence for each $k_1 \in \{1, \dots, K\}$, $\binom{K}{k_1} \left((1 - \sigma(X))^2 \right)^{k_1} (\sigma(X) (2 - \sigma(X)))^{K-k_1} (g(\sigma(X), K) - h(\sigma(X), K))$ decreases when $\sigma(X)$ increases and thus the sum decreases.

To summarize, for each $K \in \mathbb{N}^*$, $f(\sigma(X), K)$ decreases on the interval $[0, 1]$ which implies that $1/2$ is the only fixed point of $f(\sigma(X), K)$ on the interval $[0, 1]$. This completes the proof. $\square$

Proof of Lemma 5. Table 5 shows the impulses obtained to the alternative not chosen, similar to a payoff table.

Impulse balance equilibrium requires that player i 's expected impulse from X_i to Y_i is equal to his expected impulse from Y_i to X_i , $i \in N$. This yields to the following impulse balance equation: $\prod_{i \in N} \sigma_i(X_i) = \prod_{i \in N} (1 - \sigma_i(X_i))$ which completes the proof. $\square$

Player 3					
X_3				Y_3	
Player 2					
		X_2	Y_2		
Player	X_1	1, 1, 1	0, 0, 0	Player	X_1
1	Y_1	0, 0, 0	0, 0, 0	1	Y_1

Table 5: Impulses in the direction of the alternative not chosen.