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## Unique Bid Auction Games

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**Abstract** Two auction mechanisms are studied in which players compete with one another for an exogenously determined prize by independently submitting integer bids in some discrete and commonly known strategy space specified by the auctioneer. In the unique lowest (highest) bid auction game, the winner of the prize is the player who submits the lowest (highest) bid provided that this bid is unique, i.e., unmatched by other bids. Assuming a commonly known finite population of players, a non-negative cost of entry, and an option to stay out of the auction if the entry cost is deemed too high, we propose an algorithm for computing symmetric mixed-strategy equilibrium solutions to the two variants of the auction game, illustrate them, and examine their properties.

JEL Classification: C72, D44

Keywords: Unique bid auction games; Equilibrium analysis

## 1 Introduction

This paper considers two interactive decision making games called *unique bid auctions*. The study of these games has been motivated by a new type of on-line auction which has been gaining popularity in Western Europe, Australia, and the US.<sup>1</sup> The two auction games have the following simple structure. Each of  $n$  players independently decides whether to enter the auction or stay out. Players opting to enter the auction incur a fixed, non-refundable cost of entry. Subsequently, each entrant chooses a single bid from a common set of integers pre-specified by the auctioneer. In the *lowest unique bid auction* (LUBA) game, the winner is the player submitting the lowest unique bid (i.e., the lowest bid among unique bids). In the *highest unique bid auction* (HUBA) game, the winner is the player submitting the highest unique bid. The game may or may not have a winner. If there is a winner, then pays her bid and receives an exogenously determined prize with common valuation. All other  $n-1$  bidders receive nothing.

The novel feature of the LUBA and HUBA games, that sharply differentiates them from classical auctions (e.g., first-price sealed-bid auctions), is the requirement that the bid be unique in order to be considered as an eligible bid. Unique bid auction games include no exogenous lottery mechanism for breaking ties; rather, a necessary condition for winning the prize is for the winning bid to be unique (unmatched). This new feature renders the connection between the winning bid and the value of the prize tenuous. Variants of unique bid auctions have been studied both theoretically and empirically by several researchers including Raviv and Virag (2007), Östling et al. (2008), Eichberger and Vinogradov (2008), and Houba et al. (2008).

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<sup>1</sup> A considerable number of websites adopt a similar auction format to sell a variety of items, such as cars, houses, solid gold bars, gift cards, plasma televisions, MP3 players, laptop computers, and professional football tickets. Examples are Auctions4aCause.com, limbo.com, GlobalBidders.com, hammerdeal.de, uniquebidhomes.co.au, and liverpoolfc.tv. These websites use more complicated rules than the unique bid auction games studied in the current paper.

Although all of these studies maintain the key feature of the uniqueness of winning bid, they differ from one another in some key assumptions.

Studying unique bid auction games is the first step in understanding the richer structure of on-line unique bid auctions, some of them allowing several rounds of bidding until a winner is chosen, some allowing each bidder to submit (and pay for) multiple bids, and most of them requiring the bidding to proceed sequentially, updating and publicly displaying the number of bidders before some threshold number of bidders is exceeded and the auction takes place. We report equilibrium solutions of the LUBA and HUBA games as well as their following features:

1. Neither the LUBA nor HUBA games possess a symmetric equilibrium in pure strategies.
2. Any asymmetric pure-strategy equilibrium includes one bidder choosing the minimum (maximum) bid and the other bidders choosing to stay out in the LUBA (HUBA) game.
3. The support of the symmetric mixed-strategy equilibrium for the LUBA game always includes the minimum bid, and if the support includes multiple bids, then they must be consecutive.
4. The symmetric mixed-strategy equilibrium probabilities for the LUBA game are strictly decreasing in the bids.
5. The support of a symmetric mixed-strategy equilibrium for the HUBA game covers all bids that make a net payoff of winning at least as much as the equilibrium payoff.

The paper is organized as follows. Section 2 reviews the related literature. Section 3 formally presents our LUBA and HUBA games and then states several characteristics of pure- and mixed-strategy equilibria of these games. Section 4 describes our indirect approach for computing symmetric mixed-strategy equilibria. Section 5 conducts comparative static analysis by systematically changing values of the parameters. Section 6 generalizes the LUBA and HUBA

games and their equilibrium solutions to the case of incomplete information where the number of bidders is not known but only its distribution is assumed to be common knowledge. Section 7 summarizes the results.

## 2 Related Literature

Raviv and Virag (2007) considered a similar selling mechanism that they call *gambling auction*.<sup>2</sup> Their game includes a fixed number of bidders with a common valuation of an exogenously determined prize. Similarly to our LUBA game, each bidder pays an entry fee and submits a single bid. The winner is the bidder who submits the unique highest bid. Unlike the LUBA game, if such a bid does not exist, then the auctioneer repeats the auction with the same set of bidders. The winner receives the prize and pays an amount equal to her bid, whereas non-winners only pay the entry fee. The equilibrium analysis of their game is an approximation based on an assumption that the distance between the maximum allowed bid and a winning bid is negligibly small.<sup>3</sup> This assumption transforms the bidder's objective from maximizing the expected payoff into maximizing the probability of winning the prize.<sup>4</sup> Under this assumption, Raviv and Virag derived the equilibrium solution by restricting the probability of winning to be constant across bids.

Östling et al. (2008) reported an analysis of what they call LUPI (lowest unique positive integer) games. Similarly to the LUBA game, bidders in the LUPI game independently choose

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<sup>2</sup> Raviv and Virag concede that the rules of the mechanism "do not meet the traditional definitions of a lottery" (p. 3). Nevertheless, they refer to the mechanism also as lottery because under the symmetric mixed-strategy equilibrium solution the winner is chosen probabilistically. In doing so, they opt not to differentiate between lotteries in which the probabilities are determined endogenously (e.g., like in the "matching pennies" game) or exogenously with pre-determined values. Commonly, the term "gamble" is reserved to lotteries with exogenously determined probabilities (e.g., slot machines, roulette wheels).

<sup>3</sup> This assumption is built upon their observation in the field data collected from Auction4aCause.com.

<sup>4</sup> More precisely, the bidder's new problem is to maximize the probability of winning the prize multiplied by the difference between the prize and the maximum allowed bid. Since this difference is a fixed value, the bidder's problem boils down to maximizing the probability of winning the prize.

positive integers, and the player bidding the lowest unique integer wins a fixed prize of 1 and pays nothing.<sup>5</sup> Unique to their study is that their game treats the number of bidders as a random variable with a commonly known distribution. In particular, and mostly for analytical reasons, they assume that this distribution is Poisson. They report an exact equilibrium solution to this case using the theory of Poisson games of Myerson (1998, 2000), who proved its uniqueness, and described its properties.

Eichberger and Vinogradov (2008) provided the most relevant framework for LUBA games played on the Internet. They constructed analytical solutions for a variant of the LUBA game in which the number of potential bidders is fixed and commonly known, and the outside option (i.e., not entering the auction) is an element of the player's strategy space.<sup>6</sup> The major difference from other studies in the literature, which is also their major contribution, is that each bidder is allowed to submit multiple bids. This assumption renders their model closer to the lowest unique bid auctions on the Internet than any other study, but at the same time it makes the characterization of a general solution of the game complicated.

Houba et al. (2008) study a lowest unique bid auction (like LUBA) in which each bidder chooses a single bid and, as in Eichberger and Vinogradov and the present study, participation in the auction is voluntary.<sup>7</sup> Houba et al. characterized its general properties and then used three numerical methods to solve for the equilibrium solution.

Two comments on this literature are in order. First, common to the first three studies is an attempt to test their models with field data. Critical to the models of Raviv and Virag and of

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<sup>5</sup> The prize is fixed in LUPA games. This implies that as in Raviv and Virag (2007) the bidder's problem is equivalent to maximizing the probability of winning the prize.

<sup>6</sup> They call this type of auction as the least-unmatched price auction (LUPA).

<sup>7</sup> Houba et al. commented that a previous draft of the present paper assumed zero entry cost and non-voluntary participation. As we show below, these restrictions no longer hold in both the LUBA and HUBA games.

Östling et al., as to the present model, is the assumption that each bidder submits a single bid. Therefore, the number of bids is the same as the number of bidders. In contrast, unique bid auctions conducted on the Internet do not restrict the number of bids per entrant. Clearly, if a bidder submits multiple bids, the bids will necessarily differ from one another. Consequently, multiple bids by the same bidder cannot be considered independent, and field data are, in our judgment, inappropriate for testing models postulating single bids. A second equally important issue with using field data is the possibility of collusion between bidders. The model of Eichberger and Vinogradov overcomes this issue by explicitly allowing dependency of bids within a bidder while it still requires independence between bidders. However, it may be infeasible, or even impossible, to exclude cases of collusion between bidders from field data.<sup>8</sup> Therefore, the assumption of bidder independence may not be valid in field data.

Second, all the studies that allow for a single bid per player attempted to solve for symmetric mixed-strategy equilibria by what we call a *direct approach* that consists of stating a system of equations and solving them numerically. Although the direct method is straightforward in principle, it is computationally explosive. Hence, their numerical results are restricted to the very special case of a relatively small number of bidders (at most 9 bidders). To mitigate similar difficulties in the computation of the symmetric mixed-strategy equilibrium solutions to our LUBA and HUBA games, we proceed with a different *indirect approach* in which we solve the games from the viewpoint of a designated bidder, assuming that all the other bidders adhere to a symmetric mixed-strategy equilibrium and using a non-stationary Markov chain to reduce the computational burden. This considerably weakens the restriction on the number of bidders in the equilibrium solutions presented in the previous studies.

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<sup>8</sup> Collusion between bidders has been reported in the field data examined by Östling et al. Eichberger and Vinogradov also reported that their field data might have included cases that a single bidder submits multiple sets of bids using different identities.

Our paper complements contributions of the existing literature in two major ways. First, we establish several characteristics of the equilibrium solutions of both the LUBA and HUBA games. Second, we construct an algorithm to numerically solve for the symmetric mixed-strategy equilibria of these games. As shown in Section 6, not only does our indirect approach introduce a symmetric mixed-strategy equilibrium for the game with a relatively large number of bidders but it is also extendable to the case where the number of bidders is a random variable.

### 3 Theory

#### 3.1 Unique Bid Auction Games

There are  $n$  bidders who independently decide whether to enter an auction by submitting a single bid or stay out. To participate in the auction, bidders must pay a non-refundable fixed entry fee to the auctioneer. The winner is the bidder who submits the lowest (highest) bid among unique (i.e., unmatched) bids in the LUBA (HUBA) game. She must pay her bid and then receives the prize from the auctioneer. If there is no winner, every bidder receives nothing, and the prize remains in the auctioneer's hands.

We assume risk-neutrality and denote the set of identical bidders by  $I = \{1, 2, \dots, n\}$ . All bidders have the common set of strategies  $B = \{1, 2, 3, \dots\} \cup \{so\}$  where “so” stands for the decision to stay out. Each bidder independently chooses her strategy  $b_i \in B$ . We let  $\mathbf{b} = (b_i, \mathbf{b}_{-i})$  denote a pure-strategy profile of all bidders, where  $\mathbf{b}_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$  is a pure-strategy profile of all but bidder  $i$ .

Players who enter the auction by choosing a single bid incur a fixed, non-refundable entry fee of  $c$  ( $c > 0$ ). The winner receives the prize  $v$  ( $v > c$ ) and pays her winning bid. Losers get nothing. Thus, bidder  $i$ 's payoff function is given by

$$u_i(b_i, \mathbf{b}_{-i}) = \begin{cases} v - b_i - c & \text{if } b_i \text{ is the lowest (highest) unique bid in the LUBA (HUBA) game} \\ -c & \text{if } b_i \text{ is not the lowest (highest) unique bid in the LUBA (HUBA) game} \\ 0 & \text{if bidder } i \text{ stays out} \end{cases}$$

In the rest of the paper, we restrict our attention to the common set of strategies  $B \equiv \{1, 2, 3, \dots, v - c\} \cup \{so\}$ . We truncate all bids above  $v - c$  because any bid that exceeds  $v - c$  is strictly dominated by the choice of “stay out.” Also, we assume that  $v - c > 1$ . This assumption is quite natural;  $v \gg c$  in unique bid auctions played on the Internet.

### 3.2 Pure-Strategy Equilibria

In both the LUBA and HUBA games, there exists no symmetric pure-strategy equilibria; any symmetric pure-strategy profile cannot be in equilibrium because there always exists a bidder who wishes to unilaterally deviate from her strategy. Therefore, pure-strategy equilibria, if any, must be asymmetric.

Below we present several properties of asymmetric pure-strategy equilibria.

**Proposition 1** *Any asymmetric pure-strategy equilibrium for the LUBA game includes a single bidder choosing the minimum bid and the remaining bidders staying out.*

To prove this proposition, we need several lemmas.

**Lemma 1** *In any asymmetric pure-strategy equilibrium for the LUBA game at least one bidder chooses the minimum bid.*

Proof of Lemma 1 Suppose to the contrary that there exists an asymmetric pure-strategy equilibrium  $\mathbf{b} = (b_1, b_2, \dots, b_n)$  in which  $b_i \neq 1$  for all  $i \in I$ . Consider the following two cases: (i) bidder  $i$  is a winner and (ii) she is not. Suppose that bidder  $i$  is a winner. Then,  $b_i > 1$  since she participates in the auction. She wishes to switch her bid from  $b_i > 1$  to  $b_i = 1$  since she still wins the auction and  $u_i(1, \mathbf{b}_{-i}) = v - 1 - c > v - b_i - c = u_i(\mathbf{b})$  for any  $b_i > 1$ . Next, suppose that bidder

$i$  is not a winner. Then, she either stays out or chooses a higher bid than 1. Again, she has an incentive to unilaterally deviate to  $b_i = 1$  in both cases; switching from staying out assures herself a payoff of  $u_i(1, \mathbf{b}_{-i}) = v - 1 - c > 0 = u_i(\mathbf{b})$ , and switching away from  $b_i > 1$  yields her a payoff of  $u_i(1, \mathbf{b}_{-i}) = v - 1 - c > -c = u_i(\mathbf{b})$ . Hence,  $\mathbf{b}$  cannot be in equilibrium, which is a contradiction. ■

Lemma 1 is silent about how many bidders choose the minimum bid in equilibrium. The next lemma proves that only a single bidder chooses the minimum bid in equilibrium.

**Lemma 2** *In any asymmetric pure-strategy equilibrium for the LUBA game only a single bidder chooses the minimum bid.*

Proof of Lemma 2 Suppose to the contrary that there exists an asymmetric pure-strategy equilibrium with at least two bidders who choose the minimum bid of 1. As a result, each of them receives a payoff of  $-c$ . However, one of them wishes to unilaterally deviate to staying out in order to assure her a payoff of 0. ■

Now, we are ready to prove Proposition 1.

Proof of Proposition 1 Without loss of generality, assume that bidder 1 bids 1. Suppose that not all of the other bidders stay out; a subset of the other bidders enter the auction. By Lemma 2, none of them bids 1. Let  $I' = \{i \in I \mid b_i > 1\}$  denote such a subset. Then, any bidder  $j \in I'$  does not win the auction, which results in the payoff of  $u_j(b_j, \mathbf{b}_{-j}) = -c$ . For any  $I' \neq \emptyset$ , there is always some bidder  $j \in I'$  who can improve her payoff by switching to staying out. ■

It follows immediately from this proposition that the LUBA game only possesses  $n$  asymmetric pure-strategy equilibria of this type. Because a similar argument is used, we claim the following proposition without stating its proof.

**Proposition 2** *Any asymmetric pure-strategy equilibrium for the HUBA game includes only a single bidder choosing the maximum bid and the remaining bidders staying out.*

Again, the HUBA game also possesses  $n$  asymmetric pure-strategy equilibria of this type.

### 3.3 Symmetric Mixed-Strategy Equilibria

A symmetric equilibrium does not exist in pure strategies but it does exist in mixed strategies.<sup>9</sup>

Let  $p_i$  be a mixed strategy for bidder  $i$ , which is a vector  $p_i = (p_i(1), p_i(2), \dots, p_i(v-c), p_i(so))$  such that  $p_i(b) \geq 0$  for all  $b \in B$  and  $\sum_{b \in B} p_i(b) = 1$ . Let  $\mathbf{p} = (p_1, p_2, \dots, p_n) = (p_i, \mathbf{p}_{-i})$  be a profile of mixed strategies, where  $\mathbf{p}_{-i} = (p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$  is a mixed-strategy profile of all but bidder  $i$ . Given  $\mathbf{p}_{-i}$ , we denote bidder  $i$ 's (expected) payoff from playing  $p_i$  by  $u_i(\mathbf{p}) = u_i(p_i, \mathbf{p}_{-i})$  in a slight abuse of notation. A mixed-strategy equilibrium is a profile  $\mathbf{p}$  such that for each  $i \in I$ ,  $u_i(b, \mathbf{p}_{-i}) = u_i(b', \mathbf{p}_{-i})$  for all  $b, b' \in S(p_i)$ , and  $u_i(b, \mathbf{p}_{-i}) \geq u_i(b'', \mathbf{p}_{-i})$  for all  $b'' \notin S(p_i)$ , where  $S(p_i) = \{b \in B \mid p_i(b) > 0\}$ , i.e.,  $S(p_i)$  is the support of  $p_i$ .

In the symmetric mixed-strategy equilibrium, all  $n$  bidders follow the same mixed strategy, namely,  $p = (p(1), p(2), \dots, p(v-c), p(so))$  for all  $i \in I$ . The corresponding strategy profile is denoted by  $\mathbf{p} = (\underbrace{p, p, \dots, p}_n) = (p, \mathbf{p}_{-i})$ , where  $\mathbf{p}_{-i} = (\underbrace{p, p, \dots, p}_{n-1})$  is a symmetric mixed-strategy profile of all but bidder  $i$ .

#### 3.3.1 Symmetric Mixed-Strategy Equilibria for the LUBA Game

We argued earlier that symmetric equilibria cannot exist in pure strategies. This implies that symmetric equilibria must be in mixed strategies and that bidders randomize over at least two strategies in  $B$ . In the LUBA game, a bidder may wish to place a low bid to win the prize; the

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<sup>9</sup> See Dasgupta and Maskin (1986) and Becker and Damjanov (2006) for the existence of a symmetric mixed-strategy equilibrium in finite symmetric games.

lower the bid she places, the higher her payoff if she wins. Thus, when bidders randomize over their strategies, they may be inclined to place more weight on lower bids than on higher bids. In this section, we present several properties of symmetric mixed-strategy equilibria for the LUBA game, which support our intuition about the bidder's behavior.

**Proposition 3** *Suppose that  $\mathbf{p}$  is a symmetric mixed-strategy equilibrium for the LUBA game.*

- (a) *Its support  $S(p)$  includes the minimum bid.*
- (b) *If its support  $S(p)$  includes at least two bids, it contains a set of consecutive bids.*

Proof of (a) Suppose to the contrary that the minimum bid of 1 is not in  $S(p)$ . Then, bidder  $i$ 's payoff from choosing the pure strategy of bid 1 must be less than or equal to the payoff under the symmetric mixed-strategy equilibrium, i.e.,  $u_i(1, \mathbf{p}_{-i}) \leq u_i(\mathbf{p})$ . Note that the equilibrium payoff cannot exceed a payoff of  $v - 2 - c$  because the maximum possible payoff if a bidder wins is  $v - 2 - c$ . Since each bidder's probability of winning the prize is strictly less than 1 under symmetric mixed-strategy equilibrium play,  $u_i(\mathbf{p}) < v - 2 - c$  for each  $i$ . Then, bidder  $i$  wishes to unilaterally deviate to bid 1 because she can win the prize for sure and increase her payoff;  $u_i(1, \mathbf{p}_{-i}) = v - 1 - c > v - 2 - c > u_i(\mathbf{p})$ , which is a contradiction. ■

Proof of (b) Suppose to the contrary that there exists a gap in bids in  $S(p)$ . We denote this bid by  $\tilde{b} \in B$ . Since  $\tilde{b} \notin S(p)$ ,  $p(\tilde{b}) = 0$ . Let  $k \in B$  be the minimum bid in  $S(p)$  such that  $\tilde{b} < k$ . Then, for bidder  $i$ ,  $u_i(\tilde{b}, \mathbf{p}_{-i}) \leq u_i(k, \mathbf{p}_{-i}) = u_i(\mathbf{p})$ . However, bidder  $i$  wishes to unilaterally deviate to bid  $\tilde{b}$  because she can increase not only the probability of winning the prize but also the payoff when she wins. Thus,  $u_i(\tilde{b}, \mathbf{p}_{-i}) > u_i(k, \mathbf{p}_{-i}) = u_i(\mathbf{p})$ , which is a contradiction. ■

**Proposition 4** *Suppose that  $\mathbf{p}$  is a symmetric mixed-strategy equilibrium for the LUBA game.*

*Let  $b$  and  $b + 1$  be bids in  $S(p)$ . Then,  $p(b) > p(b + 1)$ .*

Proof Suppose to the contrary that  $p(b) \leq p(b+1)$ . Since  $b$  and  $b+1$  are bids in  $S(p)$ , when all the other bidders use the symmetric mixed strategy  $p$  bidder  $i$  is indifferent between choosing  $b$  and choosing  $b+1$ , i.e.,  $u_i(b, \mathbf{p}_{-i}) = u_i(b+1, \mathbf{p}_{-i})$ . We want to show that this equality no longer holds if  $p(b) \leq p(b+1)$ .

Suppose that there are exactly  $u$  bidders (out of the  $n-1$  other bidders) who do not bid less than or equal to  $b-1$  and there exists no unique bid for any bid less than or equal to  $b-1$ . First, consider the probability that bidder  $i$  wins by bidding  $b+1$ , given that all the other bidders use the symmetric mixed strategy  $p$ . She wins the prize if there is no unique bidder who chooses bid  $b$  and none of the rest bids  $b+1$ ; for example, bidder  $i$  wins the prize if three bidders bids  $b$  and none of the  $u-3$  bidders bids  $b+1$ . Given that exactly  $u$  bidders do not bid less than equal to  $b-1$  and there exists no unique bid for any bid less than or equal to  $b-1$ , bidder  $i$  wins the prize by bidding  $b+1$  with probability

$$\sum_{\substack{k=0 \\ k \neq 1}}^u \binom{u}{k} [h(b)]^k [1-h(b)]^{u-k} [1-h(b+1)]^{u-k}, \quad (1)$$

where  $h(b) = p(b)/(1 - \sum_{\beta=1}^{b-1} p(\beta))$ .  $h(b)$  is the probability of bidding  $b$ , given that a bidder does not bid less than or equal to  $b-1$ .

On the other hand, when all the other bidders use the symmetric mixed strategy  $p$ , bidder  $i$  can win the prize by bidding  $b$  with probability  $[1-h(b)]^u$ . Then, the following lemma addresses the relationship between these two probabilities.

**Lemma 3** If  $p(b) \leq p(b+1)$ ,  $[1-h(b)]^u > \sum_{\substack{k=0 \\ k \neq 1}}^u \binom{u}{k} [h(b)]^k [1-h(b)]^{u-k} [1-h(b+1)]^{u-k}$ .

Proof of Lemma 3 See Appendix. ■

We denote by  $\pi_u(b-1)$  the probability that exactly  $u$  bidders do not bid less than or equal to  $b-1$  and there exists no unique bid for any bid less than or equal to  $b-1$ . Then, bidder  $i$ 's expected payoff from bidding  $b$  is

$$u_i(b, \mathbf{p}_{-i}) = (v-b) \sum_{u=0}^{n-1} \pi_u(b-1) [1-h(b)]^u - c, \quad (2)$$

and her expected payoff from bidding  $b+1$  is

$$u_i(b+1, \mathbf{p}_{-i}) = (v-b-1) \sum_{u=0}^{n-1} \pi_u(b-1) \sum_{\substack{k=0 \\ k \neq 1}}^u \binom{u}{k} [h(b)]^k [1-h(b)]^{u-k} [1-h(b+1)]^{u-k} - c. \quad (3)$$

It follows from Lemma 3 and comparison between (2) and (3) that  $u_i(b, \mathbf{p}_{-i}) > u_i(b+1, \mathbf{p}_{-i})$ , which is a contradiction. ■

Similar properties to propositions 3 and 4 have been reported and proved by Houba et al.<sup>10</sup>

### 3.3.2 Symmetric Mixed-Strategy Equilibrium for the HUBA Game

In a sharp contrast to the LUBA game, a bidder in the HUBA game is driven by two motives that operate in *opposite* directions. To win the prize, the bidder wishes to place a high bid. However, the higher the bid she places, the lower her payoff if she wins. Therefore, one may intuitively think that bidders should choose higher bids with higher probabilities. However, the following proposition dispels this intuition. Let  $B(e) = \{b \in B \setminus \{so\} \mid v-b-c > e\}$  for any  $e$ .

**Proposition 5** *Suppose that  $\mathbf{p}$  is a symmetric mixed-strategy equilibrium for the HUBA game and the associated equilibrium payoff is  $e$ . Then, its support  $S(p)$  is equal to  $B(e)$ .*

Proof To prove this proposition, we need to show (i)  $S(p) \subseteq B(e)$  and (ii)  $B(e) \subseteq S(p)$ . We begin the proof by showing that statement (i) is true. Suppose that there is a bid  $b$  such that  $b \in S(p)$  but  $b \notin B(e)$ . Then,  $b \geq v-e-c$ . Since  $b \in S(p)$ , when all the other bidders use the

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<sup>10</sup> See Theorem 4 in Houba et al. (2008).

symmetric mixed strategy  $p$ , bidder  $i$ 's expected payoff from choosing the pure strategy of bid  $b$  must be equal to  $e$ , i.e.,  $u_i(b, \mathbf{p}_{-i}) = e$ . It also follows from  $b \in S(p)$  that bidder  $i$ 's probability of winning by choosing  $b$  must be less than 1 because  $p(b) > 0$ . Therefore,  $u_i(b, \mathbf{p}_{-i}) < v - b - c$ . Since  $b \geq v - e - c$ , we have  $e \geq v - b - c$ . Then,  $u_i(b, \mathbf{p}_{-i}) = e < v - b - c \leq e$ , which is a contradiction. Hence, statement (i) is true.

Now, show that statement (ii) is also true. Suppose that there is a bid  $b$  such that  $b \in B(e)$  but  $b \notin S(p)$ . Let  $b_{\max}^{S(p)}$  be the maximum bid in  $S(p)$ . We consider the following two cases:  $b > b_{\max}^{S(p)}$  and  $b < b_{\max}^{S(p)}$ .

Consider  $b > b_{\max}^{S(p)}$ . Since  $b \notin S(p)$ ,  $p(b) = 0$ . This implies that  $u_i(b, \mathbf{p}_{-i}) \leq e$ . If bidder  $i$  unilaterally deviates from the symmetric mixed strategy  $p$  to bidding  $b$ , she can win the prize for sure. Thus,  $u_i(b, \mathbf{p}_{-i}) = v - b - c$ . Since  $b \in B(e)$ ,  $b < v - e - c$ . Thus,  $e < v - b - c = u_i(b, \mathbf{p}_{-i}) \leq e$ , which is a contradiction.

Now, consider  $b < b_{\max}^{S(p)}$ . Let  $k \in B$  be the minimum bid such that  $k > b$  and  $k \in S(p)$ . Then,  $b < k \leq b_{\max}^{S(p)}$ . Note that when all other bidders follow the symmetric mixed-strategy  $p$ , then bidder  $i$ 's expected payoff from bidding  $b$  must be strictly smaller than her expected payoff from bidding  $k$ , i.e.,  $u_i(b, \mathbf{p}_{-i}) < u_i(k, \mathbf{p}_{-i})$ .

Suppose that  $k = b_{\max}^{S(p)}$ . When all other bidders use the symmetric mixed-strategy  $p$ , bidder  $i$ 's expected payoff from bidding  $k$  is

$$u_i(k, \mathbf{p}_{-i}) = u_i(b_{\max}^{S(p)}, \mathbf{p}_{-i}) = (v - k)[1 - p(b_{\max}^{S(p)})]^{n-1} - c. \quad (4)$$

On the other hand, bidder  $i$ 's expected payoff from bidding  $b$  is

$$u_i(b, \mathbf{p}_{-i}) = (v - b)[1 - p(b_{\max}^{S(p)})]^{n-1} + (v - b) \sum_{j=2}^{n-1} \binom{n-1}{j} [p(b_{\max}^{S(p)})]^j [1 - p(b_{\max}^{S(p)})]^{n-1-j} - c. \quad (5)$$

Hence, it follows from comparison between (4) and (5) that  $u_i(b, \mathbf{p}_{-i}) > u_i(k, \mathbf{p}_{-i})$ , which is a contradiction.

Next, suppose that  $k < b_{\max}^{S(p)}$ . Denote by  $\pi_u(k+1)$  the probability that there exists no unique bid for any bid higher than or equal to  $k+1$  and exactly  $u$  bidders (out of the  $n-1$  other bidders) do not bid higher than or equal to  $k+1$ . Then, when all other bidders play the symmetric mixed-strategy  $p$ , bidder  $i$ 's expected payoff from bidding  $k$  is

$$u_i(k, \mathbf{p}_{-i}) = (v - k)\pi_0(k+1) + (v - k) \sum_{u=1}^{n-1} \pi_u(k+1)[1 - h(k)]^u - c,$$

where  $h(k) = p(k)/(1 - \sum_{\beta=k+1}^{v-c} p(\beta))$ .  $h(k)$  is the probability of bidding  $k$  given that a bidder does not bid higher than or equal to  $k+1$ . Thus,  $\pi_u(k+1)[1 - h(k)]^u$  is the probability that there exists no unique bid for higher bids than  $k$  and none of the  $u$  bidders bids  $k$ . On the other hand, bidder  $i$ 's expected payoff from bidding  $b$  is

$$u_i(b, \mathbf{p}_{-i}) = (v - b)\pi_0(k+1) + (v - b) \sum_{u=1}^{n-1} \pi_u(k+1)\{1 - uh(k)[1 - h(k)]^{u-1}\} - c,$$

where  $uh(k)[1 - h(k)]^{u-1}$  is the probability that exactly one bidder (out of the  $u$  bidders) chooses bid  $k$ . Note that

$$[1 - h(k)]^u = 1 - uh(k)[1 - h(k)]^{u-1} \text{ for } u = 1$$

and

$$[1 - h(k)]^u < 1 - uh(k)[1 - h(k)]^{u-1} \text{ for any } u > 1.$$

Then,

$$\begin{aligned}
 u_i(k, \mathbf{p}_{-i}) &= (v - k)\pi_0(k + 1) + (v - k)\sum_{u=1}^{n-1}\pi_u(k + 1)[1 - h(k)]^u - c \\
 &< (v - k)\pi_0(k + 1) + (v - k)\sum_{u=1}^{n-1}\pi_u(k + 1)\{1 - uh(k)[1 - h(k)]^{u-1}\} - c \\
 &< (v - b)\pi_0(k + 1) + (v - b)\sum_{u=1}^{n-1}\pi_u(k + 1)\{1 - uh(k)[1 - h(k)]^{u-1}\} - c \\
 &= u_i(b, \mathbf{p}_{-i})
 \end{aligned}$$

which is a contradiction. Since both statements (i) and (ii) are true,  $S(p) = B(e)$ . ■

Proposition 5 claims that bidders assign positive probabilities to bids that result in a net payoff of winning that exceeds the equilibrium payoff  $e$ . Thus, under symmetric mixed-strategy equilibrium play, a bidder may not choose higher bids with positive probabilities. Table 1 illustrates a symmetric mixed-strategy equilibrium for the case  $n = 3$ ,  $c = 1$ , and  $v = 500$ . Since the equilibrium payoff  $e$  is 7.0798, the symmetric mixed-strategy equilibrium solution prescribes positive probabilities only to bids smaller than  $v - e - c = 491.9202$ . As shown in Table 1, the symmetric mixed-strategy equilibrium solution dictates that a bidder should randomize over all bids in  $B(e)$ , namely, all bids smaller than 491.9202. In a sharp contrast to the LUBA game where the lowest bid must be included in  $S(p)$ , the highest bid is not necessarily an element of  $S(p)$  in the HUBA game.

-- Insert Table 1 about here --

Another implication of this proposition is that the support of the symmetric mixed strategy contains a set of consecutive bids, namely, all bids in  $B(e)$ . As illustrated in Table 1, there is no gap in bids in the support of the symmetric mixed strategy (bids between 5 and 489 are not reported in Table 1 but they also are chosen with positive probabilities).

#### 4 Computing Symmetric Mixed-Strategy Equilibria: Algorithm

This section describes a procedure that uses non-stationary Markov chains to numerically compute a symmetric mixed-strategy equilibrium for the LUBA game. An algorithm for the HUBA game is omitted because it uses an almost identical procedure.

As defined earlier, we denote by  $p$  a symmetric mixed-strategy of a bidder. That is,  $p = (p(1), p(2), \dots, p(v-c), p(so))$ , where  $p(b)$  is the probability that a bidder bids  $b \in B \setminus \{so\}$  and  $p(so)$  is the probability that a bidder chooses to stay out. Let one of the  $n$  bidders, bidder  $i$ , be a designated bidder; from now on, we observe the game from this bidder's point of view. The expected payoff for this bidder for each bid  $b$  is computed and then used to solve for the probabilities  $p(1), p(2), \dots, p(v-c), p(so)$ . Assume that each of the  $n-1$  other bidders, as well as the designated bidder, independently chooses the bids or stays out according to the probabilities  $p(1), p(2), \dots, p(v-c), p(so)$ .

Suppose that the designated bidder chooses an arbitrary bid  $b \in B \setminus \{so\}$ . To determine whether  $b$  is the winning bid, the only relevant bids of the  $n-1$  other players are the ones less than or equal to  $b$ .<sup>11</sup> As bid value increases, there are fewer other bidders who do not bid yet. In other words, the number of other bidders who bid greater than  $b$ , ranging from 0 to  $n-1$ , is a non-increasing stochastic process. If exactly one of the other player bids at any lower bid than  $b$ , or any other player(s) bids  $b$ , then the designated bidder loses (denoted by the absorbing state no win, or  $NW$ ). Therefore, the bidding process can be modeled over bids as a non-stationary Markov chain. The state space of the process is denoted by  $\Omega = \{0, 1, \dots, n-1, NW\}$ . Let  $\omega \in \Omega$  be a state. There are  $|\Omega| = n+1$  possible states for any bid.

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<sup>11</sup> In the HUBA game the only relevant bids of the  $n-1$  other bidders are the ones equal to or higher than  $b$ .

Let  $\pi(0)$  be a  $1 \times (n+1)$  initial vector, whose elements are probabilities over possible states before the game starts. Denote by  $\pi_\omega(b)$  the probability of state  $\omega$  at bid  $b$ . Before the game starts, the probability of state  $\omega = n-1$  must be 1, i.e.,  $\pi_{n-1}(0) = 1$ . Therefore,

$$\pi(0) = [\pi_0(0) \quad \pi_1(0) \quad \dots \quad \pi_{n-1}(0) \quad \pi_{NW}(0)] = [0 \quad 0 \quad \dots \quad 1 \quad 0].$$

For  $b \geq 1$ , define a  $(n+1) \times (n+1)$  transition matrix as follows:

$$T(b-1, b) = \left[ \begin{array}{cccccc|cccc} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-h(b) & 0 & \dots & 0 & 0 & h(b) & 0 & 0 \\ (h(b))^2 & 0 & (1-h(b))^2 & \dots & 0 & 0 & 2h(b)(1-h(b)) & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (h(b))^{n-1} & 0 & \binom{n-1}{2} (h(b))^2 (1-h(b))^{n-3} & \dots & (1-h(b))^{n-1} & 0 & (n-1)h(b)(1-h(b))^{n-2} & 0 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \end{array} \right],$$

where  $h(b) = \frac{p(b)}{p(b) + \dots + p(v-c)} = \frac{p(b)}{1 - \sum_{\beta=1}^{b-1} p(\beta)}$ .<sup>12</sup>  $h(b)$  is the probability of bidding  $b$ , given

that a bidder does not place any lower bid than  $b-1$ . For example, the entry intersecting the 1<sup>st</sup> row and 1<sup>st</sup> column represents the probability of transition from  $\omega = 0$  at  $b-1$  to  $\omega = 0$  at  $b$ . The entry 1 intersecting the  $(n+1)^{\text{th}}$  row and the  $(n+1)^{\text{th}}$  column gives the probability of a transition from  $\omega = NW$  at  $b-1$  to  $\omega = NW$  at  $b$ . The row vector  $\pi(b-1)$  that constitutes the probability distribution over state vectors at  $b-1$  is obtained by the following matrix multiplication:

$$\pi(b-1) = \pi(0)T(0,1)T(1,2)\dots T(b-2, b-1) = \pi(b-2)T(b-2, b-1).$$

<sup>12</sup> To construct a transition matrix, all possible transitions from one state to another must be considered. However, it is impossible for some transitions to take place. The probability of such a transition is 0. Note that the use of a transition matrix is strictly for simplifying computations. No information about bidders' choices is revealed during the auction.

For example, the  $(u+1)^{\text{th}}$  element of this vector, denoted by  $\pi_u(b-1)$ , is the probability that there is no unique bid for any bid less than or equal to  $b-1$  and  $u$  bidders (out of the  $n-1$  bidders) do not bid less than or equal to  $b-1$ .

Suppose that the designated bidder bids  $b$  and  $\omega = u$  at  $b-1$ . Then, the designated bidder becomes the winner only if none of the  $u$  bidders bids  $b$ . The probability of winning by bidding  $b$  when  $\omega = u$  at  $b-1$  is  $\pi_u(b-1)[1-h(b)]^u$ . Hence, when each of the other bidders uses the symmetric mixed strategy  $p$ , the designated bidder's expected payoff from bidding  $b$  is

$$u_i(b, \mathbf{p}_{-i}) = (v-b) \sum_{u=0}^{n-1} \pi_u(b-1)[1-h(b)]^u - c.$$

To compute the symmetric mixed-strategy equilibrium, recall that the behavior of bidders who bid higher than  $b$  does not affect the payoffs of those who bid  $b$  or lower. Thus, the expected payoff from bidding  $b$  is a function of the equilibrium probabilities only through  $p(1), p(2), \dots, p(b-1)$ . To determine  $p(b)$ , the values of  $p(1), p(2), \dots, p(b-1)$  are fixed and  $p(b)$  is varied. Since  $p(1), p(2), \dots, p(b-1)$  are fixed and used by all the other bidders, the designated bidder's expected payoff from bidding  $b$  is rewritten as

$$u_i(b, p(b) | p(1), \dots, p(b-1)) = (v-b) \sum_{u=0}^{n-1} \pi_u(b-1)[1-h(b)]^u - c.$$

We denote  $u_i(b, p(b) | p(1), \dots, p(b-1))$  by  $u_i(b, p(b))$ , unless specifically stated otherwise.

Notice that  $u_i(b, p(b))$  is continuous on  $[0, p_{\max}(b)]$ , where  $p_{\max}(b) = 1 - \sum_{\beta=1}^{b-1} p(\beta)$  is the maximum feasible value of  $p(b)$ . Then, we will use the following proposition to numerically search for  $p(b)$ .

**Proposition 6** Given  $p(1), p(2), \dots, p(b-1)$ , if  $p(b) > p'(b)$ , then  $u_i(b, p(b)) < u_i(b, p'(b))$ .

Proof See Appendix. ■

The intuition behind this proposition is that  $u_i(b, p(b))$  is strictly decreasing in  $p(b)$  because the probability of a tie (i.e., losing) increases as  $p(b)$  increases.

To compute the equilibrium probabilities, we use standard characterizations of mixed-strategy equilibrium. Suppose that  $e$  is an equilibrium expected payoff of the game. Then, (i)  $u_i(b, p(b)) \leq e$  for all  $b \in B$ , (ii)  $u_i(b, p(b)) = e$  if  $p(b) > 0$ , and (iii)  $p(b) = 0$  if  $u_i(b, p(b)) < e$ . Since the value of  $e$  is unknown, the algorithm must start with an estimate of  $e$ .<sup>13</sup> For a given value of  $e$ , the associated probabilities  $p(1), \dots, p(v-c)$  are constructed sequentially through the following algorithm that starts at bid 1 and continues through bid  $v-c$ .

Step 1 Set a value of  $e$ .

Step 2 Consider bid  $b$ . Given  $p(1), \dots, p(b-1)$ , compute  $u_i(b, 0)$ .

a. If  $u_i(b, 0) \leq e$ , then keep  $p(b) = 0$ . If  $b < v-c$ , increase  $b$  by 1 unit and repeat Step

2. Otherwise, compute  $p(so) = 1 - \sum_{\beta=1}^{v-c} p(\beta)$  and go to Step 3.

b. If  $u_i(b, 0) > e$ , evaluate  $u_i(b, p_{\max}(b))$ .

i. If  $u_i(b, p_{\max}(b)) \leq e$ , then, there exists  $p(b)$  ( $0 < p(b) \leq p_{\max}(b)$ ) such that

$u_i(b, p(b)) = e$  since  $u_i(b, p(b))$  is continuous on  $[0, p_{\max}(b)]$  and strictly

decreasing in  $p(b)$  by Proposition 6. If  $b < v-c$ , then increase  $b$  by 1 unit

and repeat Step 2. Otherwise, compute  $p(so) = 1 - \sum_{\beta=1}^{v-c} p(\beta)$  and go to Step

3.

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<sup>13</sup> Lower and upper bounds for  $e$  are  $-c$  and  $v-1-c$ , respectively. If the value of  $e$  is set too high, the sum of the associated probabilities may be much smaller than 1. On the other hand, if the value of  $e$  is set too low, no equilibrium solution may exist.

- ii. If  $u_i(b, p_{\max}(b)) > e$ , then the game has no solution for the given value of  $e$ .

Terminate the algorithm. Return to Step 1, increase  $e$  because  $e$  is a lower bound, and repeat the algorithm.

Step 3 There are three possible scenarios:

- a. If  $e > 0$  and  $p(so) > 0$ , then there is no solution. Go to Step 1, decrease  $e$  because it is an upper bound, and repeat the algorithm.
- b. If  $e > 0$  and  $p(so) = 0$ , then compute  $1 - \sum_{\beta=1}^{v-c} p(\beta)$ . If  $1 - \sum_{\beta=1}^{v-c} p(\beta) > \varepsilon$ , where  $\varepsilon$  specifies how close is the sum of the probabilities to 1, then return to Step 1, decrease  $e$ , and repeat the algorithm. Otherwise,  $p = (p(1), \dots, p(v-c), p(so))$  are the equilibrium probabilities.
- c. If  $e = 0$ ,  $p(1), \dots, p(v-c), p(so)$  are the equilibrium probabilities.

To verify that our algorithm accurately generates symmetric mixed-strategy equilibria, we consider the same numerical examples reported in Houba et al. (2008), whose game is most closely related to the LUBA game. The only difference between their game and our LUBA game is in the strategy space; Houba et al. assume that the smallest bid is zero whereas we assume that it is 1. We modified our algorithm by including bid zero in the strategy space.

Table 2 reports the symmetric mixed-strategy equilibria and their corresponding equilibrium payoffs for various values of the prize  $v$  when  $n=3$  (upper panel) and  $n=9$  (lower panel). In both cases,  $c=1$ . Our algorithm reproduces the results of Houba et al. to the fourth decimal point. The lower panel of Table 2 shows clearly that when  $v \gg c$ , the mixed-strategy equilibrium solution converges as  $v$  increases.

-- Insert Table 2 about here --

## 5 Numerical Analysis

Section 5 exhibits six figures in order to illustrate the equilibrium solutions for the LUBA and HUBA games. For each of these games, we present three figures to examine the effect of the number of players  $n$ , prize  $v$ , and entry cost  $c$ .

Figure 1 exhibits in three panels the effect of the number of bidders. In all three cases, for  $n=10, 30$ , and  $50$ , the prize is held fixed at  $v=100$  and entry cost at  $c=1$ . Because the value of  $c$  is relatively low, the probability of staying out is 0 in all three cases. As  $n$  increases, the support of the mixed strategy increases too. In agreement with Propositions 3 and 4, the bids are seen to be consecutive and the bid probabilities decrease in the value of  $b$ . Figure 2 displays symmetric mixed-strategy equilibrium solutions for the HUBA games and the same parameter values as in Fig. 1. In all three panels for  $n=10, 30$ , and  $50$   $B(e) = \{1, 2, \dots, 97\}$  (see Proposition 5). The probability of staying out  $p(so)$  increases in  $n$  and, correspondingly,  $B(e) = \{1, 2, \dots, 97\}$  becomes more “flat” as  $n$  increases.

-- Insert Figs. 1 and 2 about here --

Figure 3 exhibits solutions for  $v=10, 50$ , and  $90$  when  $n=30$  and  $c=1$ . When  $v=10$ , the equilibrium solution results in a relatively high probability of staying out, namely,  $p(so) = 0.7957$ . Beyond some threshold value of  $v$   $p(so) = 0$ .<sup>14</sup> The effect of  $v$  is similar to the effect of  $n$  in the LUBA game (Fig. 1); as  $v$  increases, the support of the mixed strategy increases. Figure 4 displays similar equilibrium solutions for three HUBA games for the same parameter values as in Fig. 3. A notable feature of the mixed-strategy equilibrium solution for the HUBA games is that the equilibrium probability distribution shifts horizontally towards higher bids as  $v$  increases. The probabilities of staying out decrease slightly but monotonically as  $v$  increases.

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<sup>14</sup>  $p(so)=0.0047$  when  $v=37$  while  $p(so)=0$  when  $v=38$ .

-- Insert Figs. 3 and 4 about here --

The last two figures in this section illustrate the effects of changes in the entry cost  $c$  on the equilibrium solution. In the LUBA game (Fig. 5),  $p(so) = 0$  increases from 0 if  $c=2$  to 0.2456 if  $c=4$  and then to 0.5771 if  $c=7$ . In all three cases,  $n=30$  and  $v=100$ . The same effect holds in the three HUBA games depicted in Fig. 6. Moreover, the support  $B(e) = \{1,2,...,97\}$  of the mixed strategy shrinks if  $c$  increases when  $n$  and  $v$  are held constant.

-- Insert Figs. 5 and 6 about here --

## 6 Extension to Random Number of Bidders

The unique bid auction games have been studied under the assumption that  $n$  is fixed and commonly known. The exception is Östling et al. (2008), who restricted their analysis to the special case where  $n$  is a random variable with a Poisson distribution. Under most general circumstances, the exact value of  $n$  may not be known with certainty. We propose capturing this uncertainty with the assumption that  $n$  is a random variable with a commonly known distribution.

When considering games with an uncertain number of bidders, caution should be exercised in distinguishing between the probability distribution of the number of bidders perceived by an outside observer and the probability distribution perceived by a bidder who participates in the game.<sup>15</sup> To illustrate this distinction, consider a LUBA game in which the number of bidders is either 100 or 150 with equal probability. An observer looking at the game from the outside would conclude that the expected number of bidders is 125. A bidder who participates in the game would conclude that she is 1.5 times as likely to interact with 149 than with 99 other bidders. She then updates the conditional probability of interacting with 99 other bidders to  $2/5$

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<sup>15</sup> For the importance of this distinction, see Cooper (1981) and Myerson (1998).

and the conditional probability of interacting with 149 other bidders to  $3/5$ . From her perspective, the expected number of other bidders she plays with is  $99 \times 2/5 + 149 \times 3/5 = 129$ . Therefore, including herself, she expects 130 (rather than 125) bidders to participate in the game.

A major advantage of the present computational approach is that the modifications of the algorithm in Section 3 that take care of this distinction are minimal; one only needs to modify the initial vector and transition probability matrix. Recall that the elements of the initial vector specify probabilities over possible states before the game starts. When the number of bidders,  $n$ , is fixed and known, then the  $1 \times (n+1)$  initial vector takes the following form:

$$\pi(0) = [\pi_0(0) \quad \pi_1(0) \quad \dots \quad \pi_{n-1}(0) \quad \pi_{NW}(0)] = [0 \quad 0 \quad \dots \quad 1 \quad 0].$$

The probability of 1 is assigned to  $\pi_{n-1}(0)$  and all the other elements take the value of 0.

Suppose that  $n$  is either equal to  $n_L$  with probability  $\Pr(n = n_L)$  or  $n_H$  with probability  $\Pr(n = n_H)$ , where  $n_L < n_H$ ,  $\Pr(n = n_L) + \Pr(n = n_H) = 1$ , and that this distribution is common knowledge. A set of possible states is denoted by  $\hat{\Omega} = \{0, 1, \dots, n_L - 1, \dots, n_H - 1, NW\}$  and a state by  $\hat{\omega} \in \hat{\Omega}$ . There are  $|\hat{\Omega}| = n_H + 1$  possible states for any bid. Then, we denote by  $\hat{\pi}(0)$  a  $1 \times |\hat{\Omega}|$  initial vector whose elements are probabilities over possible states before the game starts. To construct this vector, one must compute probabilities of the following two possible states: (i)  $n_L - 1$  other bidders are in the game and (ii)  $n_H - 1$  other bidders are in the game. Denote by  $\Pr(n = n_L | In)$  the conditional probability of  $n = n_L$ , given that a bidder is one of the participants in the game. Then,

$$\begin{aligned}
 \Pr(n = n_L | In) &= \frac{\Pr(n = n_L \cap In)}{\Pr(In)} \\
 &= \frac{\Pr(n = n_L \cap In)}{\Pr(n = n_L \cap In) + \Pr(n = n_H \cap In)} \\
 &= \frac{\Pr(n = n_L) \Pr(In | n = n_L)}{\Pr(n = n_L) \Pr(In | n = n_L) + \Pr(n = n_H) \Pr(In | n = n_H)}.
 \end{aligned}$$

We assume that the auctioneer assigns bidders to auctions in a way that results in “size-biased” sampling (Stein and Dattero, 1985), which is discussed above. This results in the assumption  $\Pr(In | n = n_H) = (n_H / n_L) \Pr(In | n = n_L)$ . Then,

$$\begin{aligned}
 \Pr(n = n_L | In) &= \frac{\Pr(n = n_L) \Pr(In | n = n_L)}{\Pr(n = n_L) \Pr(In | n = n_L) + (n_H / n_L) \Pr(n = n_H) \Pr(In | n = n_L)} \\
 &= \frac{\Pr(n = n_L)}{\Pr(n = n_L) + (n_H / n_L) \Pr(n = n_H)} \\
 &= \frac{n_L \Pr(n = n_L)}{n_L \Pr(n = n_L) + n_H \Pr(n = n_H)}.
 \end{aligned}$$

Similarly, the conditional probability of  $n = n_H$ , given that a bidder is one of the participants in the game,  $\Pr(n = n_H | In)$  is computed from

$$\Pr(n = n_H | In) = 1 - \Pr(n = n_L | In) = \frac{n_H \Pr(n = n_H)}{n_L \Pr(n = n_L) + n_H \Pr(n = n_H)}.$$

In the initial vector  $\hat{\pi}(0)$ ,  $\Pr(n = n_L | In)$  is assigned to the state in which  $n_L - 1$  other bidders are in the game, while  $\Pr(n = n_H | In)$  is assigned to the state in which  $n_H - 1$  other bidders are in the game. In other words,

$$\begin{aligned}
 \hat{\pi}(0) &= [\hat{\pi}_0(0) \quad \hat{\pi}_1(0) \quad \dots \quad \hat{\pi}_{n_L-1} \quad \dots \quad \hat{\pi}_{n_H-1} \quad \hat{\pi}_{NW}(0)] \\
 &= [0 \quad 0 \quad \dots \quad \Pr(n = n_L | In) \quad \dots \quad \Pr(n = n_H | In) \quad 0].
 \end{aligned}$$

A  $|\hat{\Omega}| \times |\hat{\Omega}|$  transition matrix whose elements are probabilities of transition from one state to another must be defined. Then, the same algorithm described in Section 3.1 applies here with these new initial vector and transition matrix.

Table 3 presents symmetric mixed-strategy equilibria for the following four cases in the LUBA game: (i)  $n = 20$ , (ii)  $\Pr(n_L=15)=0.625$  and  $\Pr(n_H=25)=0.375$ , (iii)  $\Pr(n_L=10)=0.75$  and  $\Pr(n_H=30)=0.25$ , and (iv)  $\Pr(n_L=5)=0.875$  and  $\Pr(n_H=35)=0.125$ . Each bidder in all these four cases expects the same number of other bidders; any bidder in the first case knows that there are exactly 19 other bidders whereas for any bidder who participates in the other three cases the *expected* number of other bidders is 19. For each of these cases,  $v=50$  and  $c=1$ . Table 4 shows similar results for the same parameter values in the HUBA game

-- Insert Tables 3 and 4 about here --

Table 3 displays results that give rise to several conjectures. In the present example, if  $n$  assumes one of two values  $n_L$  and  $n_H$  and the expected number of other bidders is held fixed at  $n-1=19$ , then the expected payoff for the LUBA game increases dramatically from 0.1926 when  $n=20$  to 1.8165 when  $p(n_L=5)=0.875$  and  $p(n_H=35)=0.125$ . In the four HUBA games (Table 4), as the uncertainty about  $n$  increases the probability of staying out decreases from 0.7906 to 0.4376.

## 7 Conclusion

Unique bid auctions implemented on the Internet, whether of the LUBA or HUBA type, introduce a new format of auctions. There are by now hundreds of websites that conduct auctions in which the winning bid must simultaneously satisfy two criteria, namely, it must be the lowest (in the LUBA game) or highest (in the HUBA game) among all the bids, and it must be unmatched by the other bids; see examples from Sweden, Holland, and Germany in the papers by Östling et al. (2008), Houba, et al. (2008), and Eichberger and Vinogradov (2008) and dozens of British, Australian, and US websites that advertise on the Internet. Our contribution in this paper is a new algorithm for computing symmetric mixed-strategy equilibria, which models the

auction as a non-stationary Markov chain rather than approaching its numerical solution as a combinatorial problem. Our algorithm has several attractive properties. It is readily applicable to LUBA and HUBA games. It can handle exogenous as well as endogenous entry. With a minor modification, it can relax the assumption that the number of bidders is fixed. Most importantly, it is not subjected to the same computational difficulties that have plagued the numerical solutions in the previous studies.

However, the algorithm is not without limitations. It is presently restricted to unique bid auctions with homogenous players, single bids, and a single round of play. It generates a symmetric mixed-strategy solution but it is still an open question whether the solution is unique. As such, the algorithm has not been intended to capture the richer format of on-line unique bid auctions nor is it applicable to field data. We view it as yet another step, complementary to the results of Raviv and Virag, Östling et al., and Houba, et al., and Eichberger and Vinogradov, and hope that collectively these studies will contribute to a better understanding of this new auction format.

### **Acknowledgements**

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## Appendix

### Proof of Lemma 3

Proof: Suppose that  $p(b) \leq p(b+1)$ . Parts in equation (1) can be expressed as follows:

$$[h(b)]^k = \left( \frac{p(b)}{1 - \sum_{\beta=1}^{b-1} p(\beta)} \right)^k,$$

$$[1 - h(b)]^{u-k} = \left( \frac{1 - \sum_{\beta=1}^b p(\beta)}{1 - \sum_{\beta=1}^{b-1} p(\beta)} \right)^{u-k},$$

$$[1 - h(b+1)]^{u-k} = \left( \frac{1 - \sum_{\beta=1}^{b+1} p(\beta)}{1 - \sum_{\beta=1}^b p(\beta)} \right)^{u-k}.$$

Then,

$$\begin{aligned} & \sum_{\substack{k=0 \\ k \neq 1}}^u \binom{u}{k} [h(b)]^k [1 - h(b)]^{u-k} [1 - h(b+1)]^{u-k} \\ &= \left( \frac{1}{1 - \sum_{\beta=1}^{b-1} p(\beta)} \right)^u \sum_{\substack{k=0 \\ k \neq 1}}^u \binom{u}{k} [p(b)]^k \left[ 1 - \sum_{\beta=1}^{b+1} p(\beta) \right]^{u-k} \\ &= \left( \frac{1}{1 - \sum_{\beta=1}^{b-1} p(\beta)} \right)^u \left\{ \left[ 1 - \sum_{\substack{\beta=1 \\ \beta \neq b}}^{b+1} p(\beta) \right]^u - \binom{u}{1} p(b) \left[ 1 - \sum_{\beta=1}^{b+1} p(\beta) \right]^{u-1} \right\} \\ &< \left( \frac{1}{1 - \sum_{\beta=1}^{b-1} p(\beta)} \right)^u \left[ 1 - \sum_{\substack{\beta=1 \\ \beta \neq b}}^{b+1} p(\beta) \right]^u \end{aligned}$$

The second equality results from the binomial theorem. Since  $p(b) \leq p(b+1)$ ,

$$\left( \frac{1}{1 - \sum_{\beta=1}^{b-1} p(\beta)} \right)^u \left[ 1 - \sum_{\substack{\beta=1 \\ \beta \neq b}}^{b+1} p(\beta) \right]^u \leq \left( \frac{1}{1 - \sum_{\beta=1}^{b-1} p(\beta)} \right)^u \left[ 1 - \sum_{\beta=1}^b p(\beta) \right]^u = [1 - h(b)]^u. \blacksquare$$

**Proof of Proposition 6**

Proof: Once  $p(1), p(2), \dots, p(b-1)$  are fixed, all the components of  $\pi(b-1)$ , in other words,

$\pi_u(b-1)$  for all  $u$ , are determined. Since  $h(b) = \frac{p(b)}{1 - \sum_{\beta=1}^{b-1} p(\beta)}$ ,  $p(b) > p'(b)$  implies

$u_i(b, p(b)) < u_i(b, p'(b))$ . ■

Table 1: Symmetric mixed-strategy equilibria of the HUBA game

for  $n = 3$ ,  $c = 1$ , and  $v = 500$ .

| Bid                        | Equilibrium Probability | Cumulative Equilibrium Probability |
|----------------------------|-------------------------|------------------------------------|
| 1                          | 0.0062                  | 0.0062                             |
| 2                          | 0.0024                  | 0.0086                             |
| 3                          | 0.0018                  | 0.0105                             |
| 4                          | 0.0015                  | 0.0120                             |
| ...                        | ...                     | ...                                |
| 490                        | 0.0501                  | 0.9475                             |
| 491                        | 0.0525                  | 1                                  |
| 492                        | 0                       | 1                                  |
| 493                        | 0                       | 1                                  |
| 494                        | 0                       | 1                                  |
| 495                        | 0                       | 1                                  |
| 496                        | 0                       | 1                                  |
| 497                        | 0                       | 1                                  |
| 498                        | 0                       | 1                                  |
| 499                        | 0                       | 1                                  |
| so                         | 0                       | 1                                  |
| Equilibrium Payoff ( $e$ ) | 7.0798                  |                                    |

Table 2: Symmetric mixed-strategy equilibria for different values of the prize  $v$  $n = 3$  and  $c = 1$ 

| Prize<br>Value ( $v$ ) | Bid    |        |        |        |        | Equilibrium<br>Payoff ( $e$ ) | Sum<br>Payoff |
|------------------------|--------|--------|--------|--------|--------|-------------------------------|---------------|
|                        | $so$   | 0      | 1      | 2      | 3      |                               |               |
| 3                      | 0.5669 | 0.4226 | 0.0105 | 0      | 0      | 0                             | 0             |
| 4                      | 0.2887 | 0.5000 | 0.2113 | 0      | 0      | 0                             | 0             |
| 5                      | 0      | 0.5191 | 0.3407 | 0.1403 | 0      | 0.1565                        | 0.4694        |
| 6                      | 0      | 0.5093 | 0.3189 | 0.1717 | 0      | 0.4445                        | 1.3336        |
| 10                     | 0      | 0.4903 | 0.2902 | 0.2154 | 0.0041 | 1.5974                        | 4.7921        |
| 15                     | 0      | 0.4801 | 0.2769 | 0.1750 | 0.0680 | 3.0539                        | 9.1617        |
| 25                     | 0      | 0.4713 | 0.2659 | 0.1577 | 0.1051 | 5.9885*                       | 17.9656*      |

 $n = 9$  and  $c = 1$ 

| Prize<br>Value ( $v$ ) | Bid    |        |        |        |        |        |        | Equilibrium<br>Payoff ( $e$ ) |
|------------------------|--------|--------|--------|--------|--------|--------|--------|-------------------------------|
|                        | 0      | 1      | 2      | 3      | 4      | 5      | 6      |                               |
| 50                     | 0.2559 | 0.2384 | 0.2112 | 0.1651 | 0.0978 | 0.0314 | 0.0002 | 3.6999*                       |
| 100                    | 0.2536 | 0.2365 | 0.2099 | 0.1646 | 0.0989 | 0.0339 | 0.0025 | 8.6289*                       |
| 200                    | 0.2526 | 0.2357 | 0.2092 | 0.1644 | 0.0995 | 0.0351 | 0.0036 | 1.4850*                       |
| 400                    | 0.2520 | 0.2352 | 0.2089 | 0.1643 | 0.0998 | 0.0357 | 0.0041 | 3.1932*                       |

\* Not provided by Houba et al. (2008)

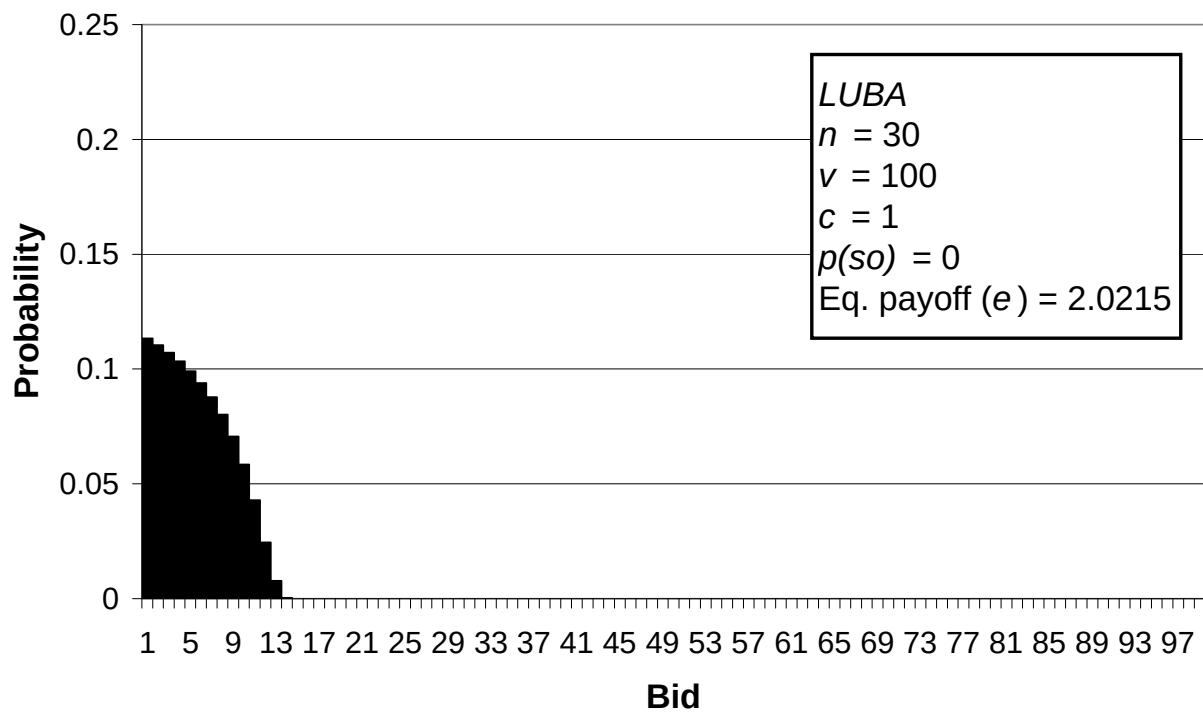
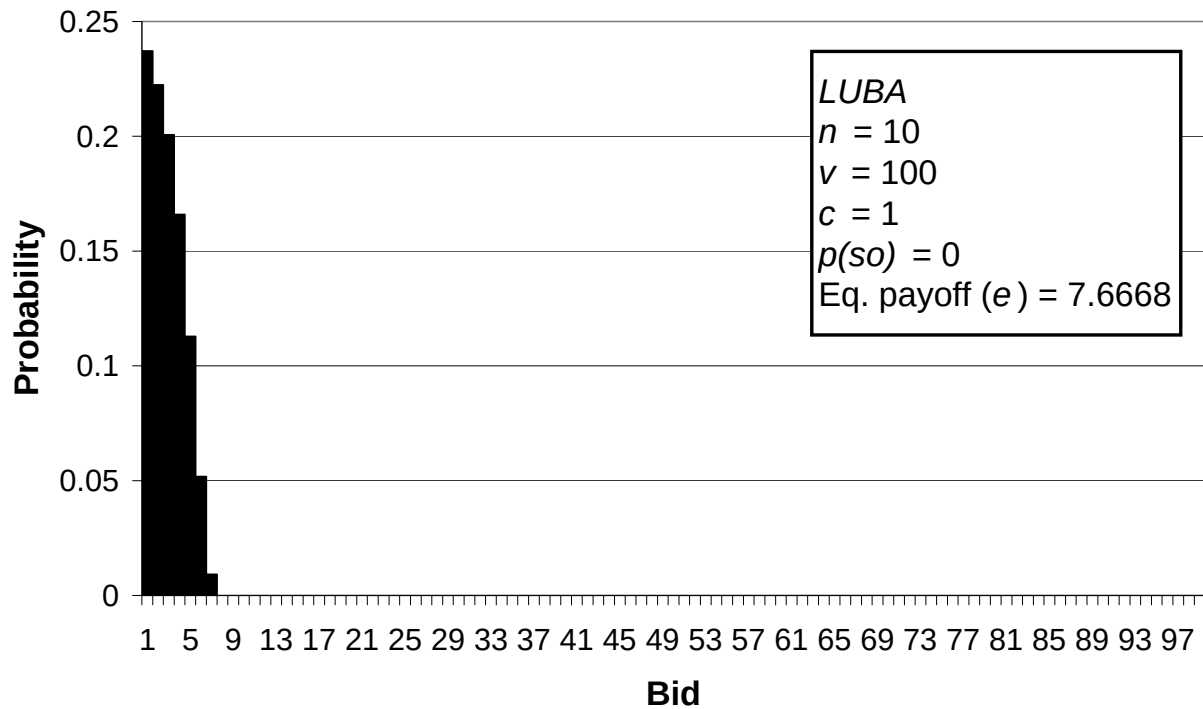
Table 3: Symmetric mixed-strategy equilibrium probabilities for the LUBA game under uncertainty for  $v = 30$  and  $c = 1$ .

| Bid                       | $n = 20$ | $\Pr(n_L = 15) = 0.625$<br>&<br>$\Pr(n_H = 25) = 0.375$ | $\Pr(n_L = 10) = 0.75$<br>&<br>$\Pr(n_H = 30) = 0.25$ | $\Pr(n_L = 5) = 0.875$<br>&<br>$\Pr(n_H = 35) = 0.125$ |
|---------------------------|----------|---------------------------------------------------------|-------------------------------------------------------|--------------------------------------------------------|
|                           |          |                                                         |                                                       |                                                        |
| 1                         | 0.1546   | 0.1677                                                  | 0.2155                                                | 0.3361                                                 |
| 2                         | 0.1486   | 0.1579                                                  | 0.1935                                                | 0.2707                                                 |
| 3                         | 0.1416   | 0.1462                                                  | 0.1622                                                | 0.1395                                                 |
| 4                         | 0.1329   | 0.1323                                                  | 0.1252                                                | 0.0841                                                 |
| 5                         | 0.1221   | 0.1165                                                  | 0.0965                                                | 0.0646                                                 |
| 6                         | 0.1080   | 0.0990                                                  | 0.0770                                                | 0.0498                                                 |
| 7                         | 0.0890   | 0.0799                                                  | 0.0602                                                | 0.0345                                                 |
| 8                         | 0.0637   | 0.0581                                                  | 0.0423                                                | 0.0176                                                 |
| 9                         | 0.0329   | 0.0330                                                  | 0.0225                                                | 0.0031                                                 |
| 10                        | 0.0065   | 0.0093                                                  | 0.0051                                                | 0                                                      |
| 11 to 29                  | 0        | 0                                                       | 0                                                     | 0                                                      |
| so                        | 0        | 0                                                       | 0                                                     | 0                                                      |
| Equilibrium<br>Payoff (e) | 0.1926   | 0.2859                                                  | 0.6447                                                | 1.8165                                                 |

Table 4: Symmetric mixed-strategy equilibrium probabilities for the HUBA game under uncertainty for  $v = 30$  and  $c = 1$ .

| Bid                       | $n = 20$ | $\Pr(n_L = 15) = 0.625$<br>&<br>$\Pr(n_H = 25) = 0.375$ | $\Pr(n_L = 10) = 0.75$<br>&<br>$\Pr(n_H = 30) = 0.25$ | $\Pr(n_L = 5) = 0.875$<br>&<br>$\Pr(n_H = 35) = 0.125$ |
|---------------------------|----------|---------------------------------------------------------|-------------------------------------------------------|--------------------------------------------------------|
| 1                         | 0.0017   | 0.0019                                                  | 0.0029                                                | 0.0050                                                 |
| 2                         | 0.0017   | 0.0020                                                  | 0.0029                                                | 0.0053                                                 |
| 3                         | 0.0018   | 0.0021                                                  | 0.0031                                                | 0.0055                                                 |
| 4                         | 0.0019   | 0.0022                                                  | 0.0032                                                | 0.0057                                                 |
| 5                         | 0.0019   | 0.0022                                                  | 0.0033                                                | 0.0059                                                 |
| 6                         | 0.0021   | 0.0023                                                  | 0.0035                                                | 0.0062                                                 |
| 7                         | 0.0021   | 0.0024                                                  | 0.0036                                                | 0.0064                                                 |
| 8                         | 0.0022   | 0.0026                                                  | 0.0038                                                | 0.0067                                                 |
| 9                         | 0.0024   | 0.0027                                                  | 0.0040                                                | 0.0070                                                 |
| 10                        | 0.0025   | 0.0029                                                  | 0.0042                                                | 0.0073                                                 |
| 11                        | 0.0026   | 0.0030                                                  | 0.0044                                                | 0.0076                                                 |
| 12                        | 0.0028   | 0.0032                                                  | 0.0046                                                | 0.0080                                                 |
| 13                        | 0.0030   | 0.0034                                                  | 0.0049                                                | 0.0083                                                 |
| 14                        | 0.0032   | 0.0036                                                  | 0.0052                                                | 0.0086                                                 |
| 15                        | 0.0035   | 0.0039                                                  | 0.0055                                                | 0.0090                                                 |
| 16                        | 0.0037   | 0.0042                                                  | 0.0059                                                | 0.0094                                                 |
| 17                        | 0.0041   | 0.0045                                                  | 0.0064                                                | 0.0098                                                 |
| 18                        | 0.0045   | 0.0050                                                  | 0.0069                                                | 0.0104                                                 |
| 19                        | 0.0049   | 0.0055                                                  | 0.0075                                                | 0.0117                                                 |
| 20                        | 0.0056   | 0.0061                                                  | 0.0083                                                | 0.0149                                                 |
| 21                        | 0.0063   | 0.0070                                                  | 0.0094                                                | 0.0215                                                 |
| 22                        | 0.0075   | 0.0082                                                  | 0.0113                                                | 0.0311                                                 |
| 23                        | 0.0093   | 0.0103                                                  | 0.0149                                                | 0.0419                                                 |
| 24                        | 0.0128   | 0.0142                                                  | 0.0213                                                | 0.0529                                                 |
| 25                        | 0.0187   | 0.0208                                                  | 0.0299                                                | 0.0636                                                 |
| 26                        | 0.0267   | 0.0290                                                  | 0.0383                                                | 0.0723                                                 |
| 27                        | 0.0340   | 0.0359                                                  | 0.0434                                                | 0.0714                                                 |
| 28                        | 0.0358   | 0.0367                                                  | 0.0399                                                | 0.0489                                                 |
| 29                        | 0        | 0                                                       | 0                                                     | 0                                                      |
| so                        | 0.7906   | 0.7725                                                  | 0.6976                                                | 0.4376                                                 |
| Equilibrium<br>Payoff (e) | 0        | 0                                                       | 0                                                     | 0                                                      |

Figure 1: Symmetric mixed-strategy equilibria for the LUBA game for  $n=10, 30$ , and  $50$ ,  $v = 100$ , and  $c = 1$ .



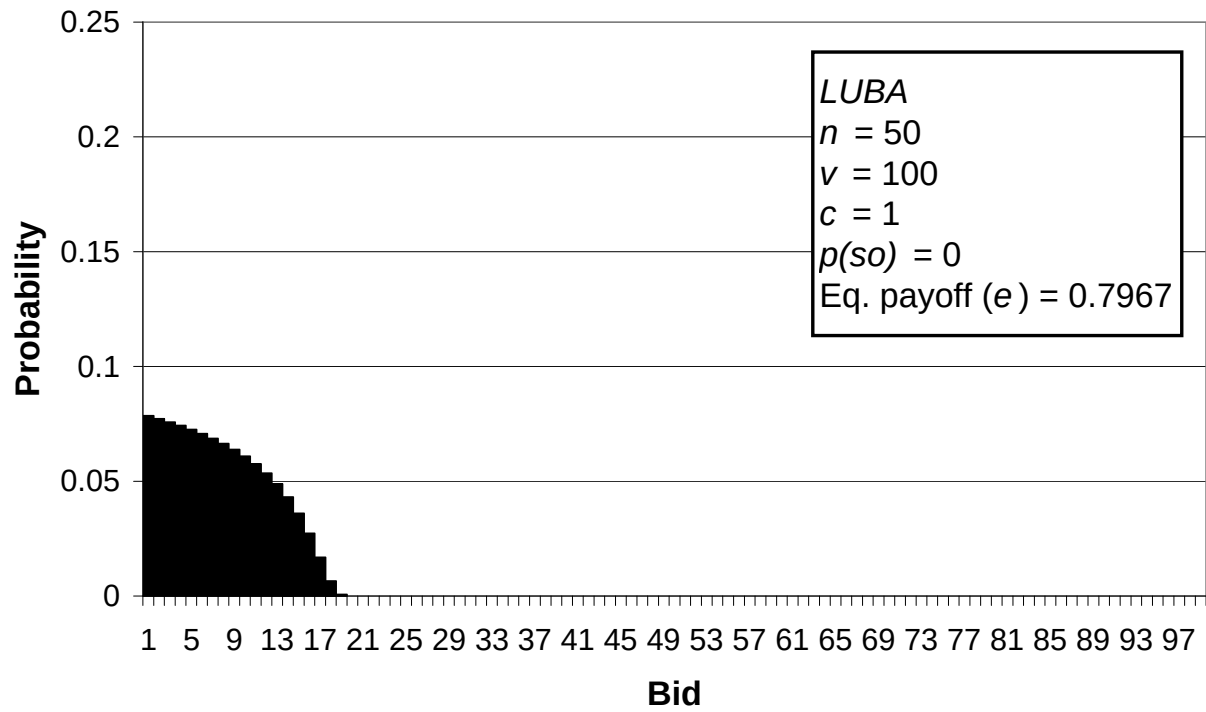
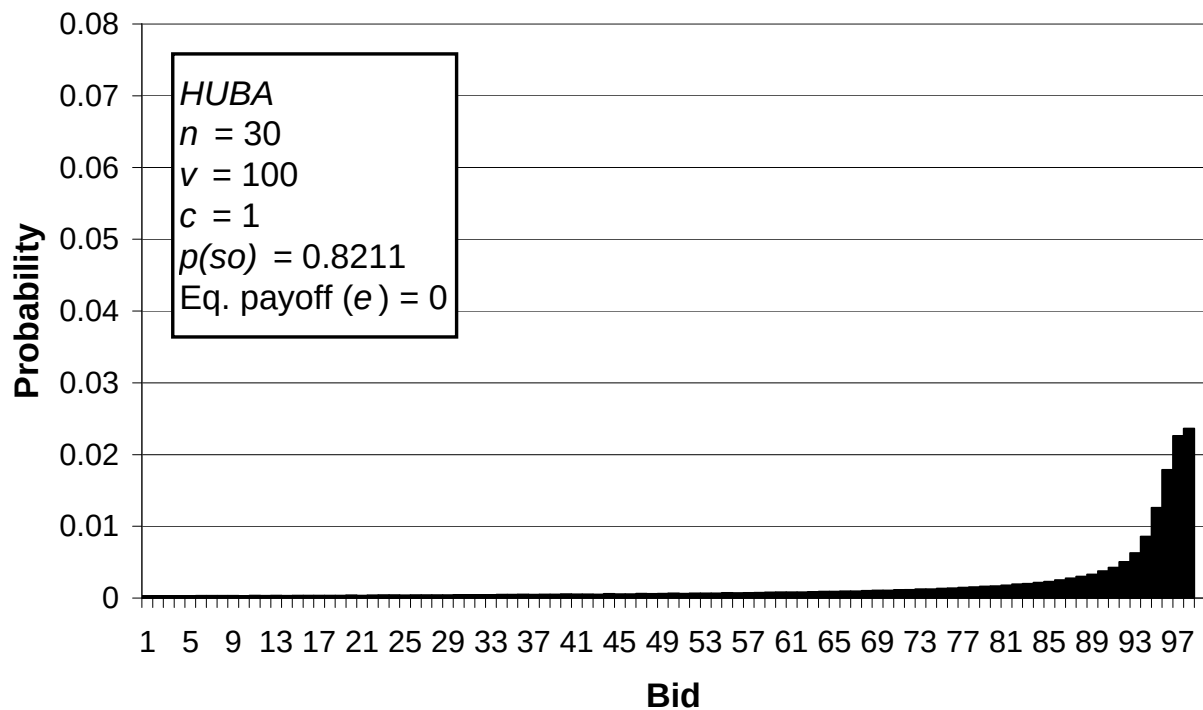
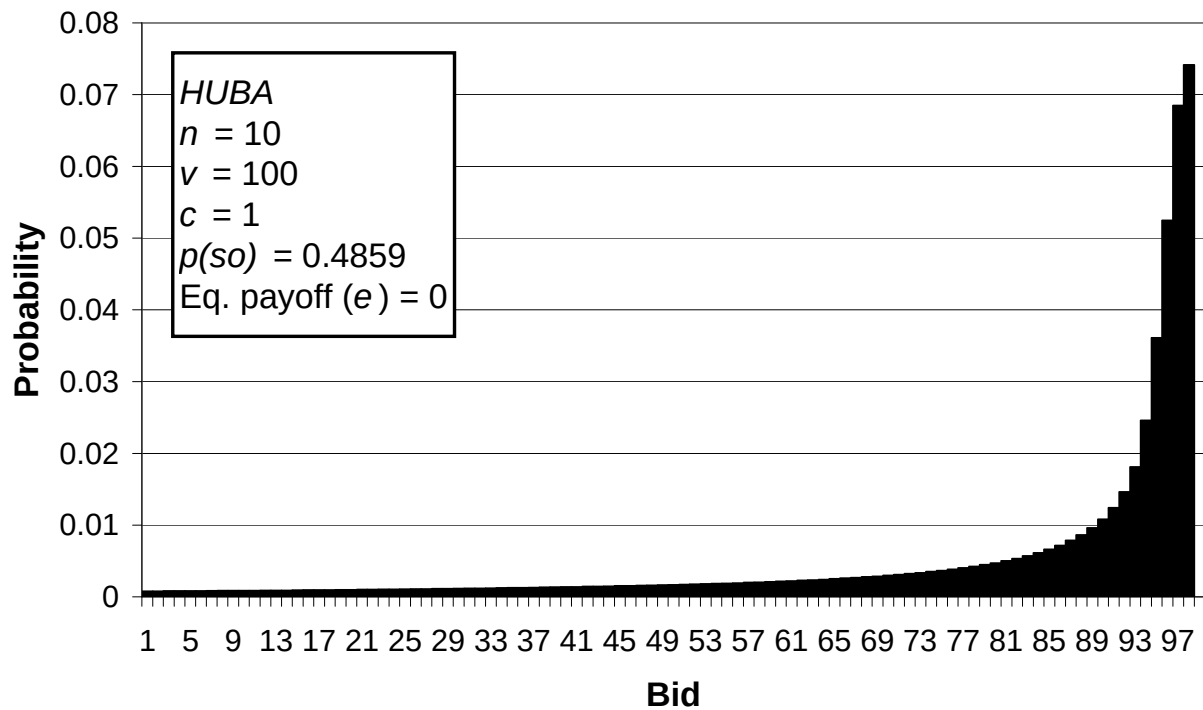


Figure 2: Symmetric mixed-strategy equilibria for the HUBA game for  $n=10, 30$ , and  $50$ ,  $v = 100$ , and  $c = 1$ .



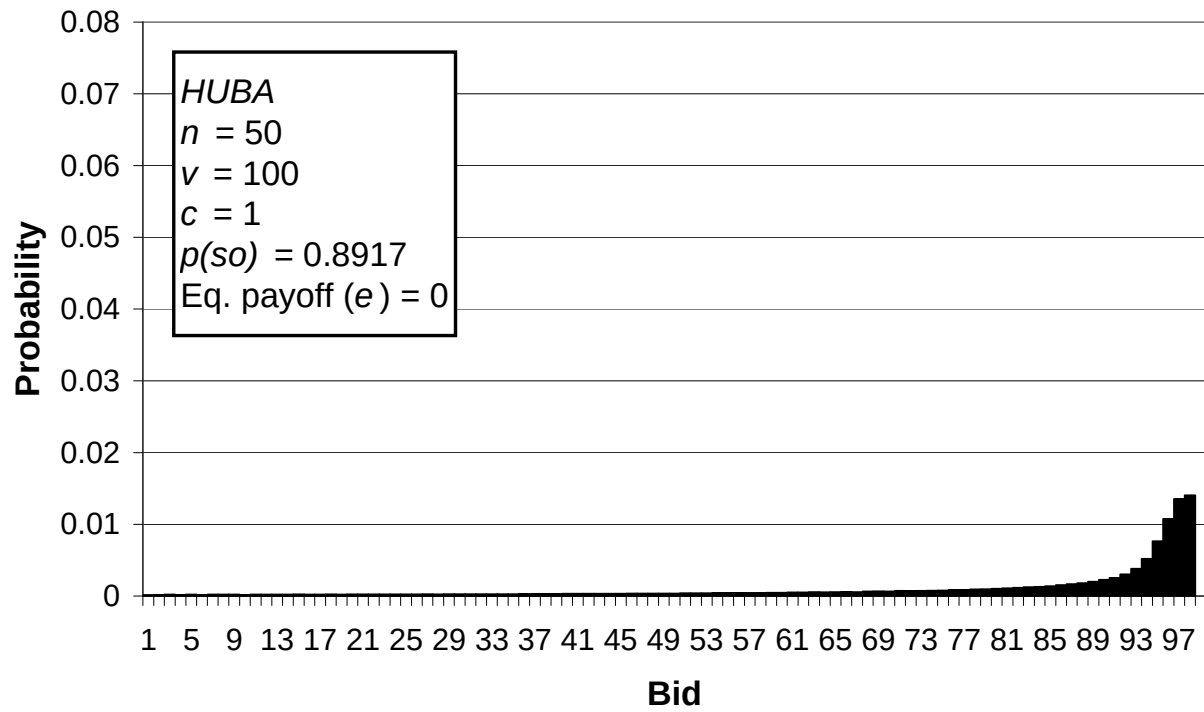
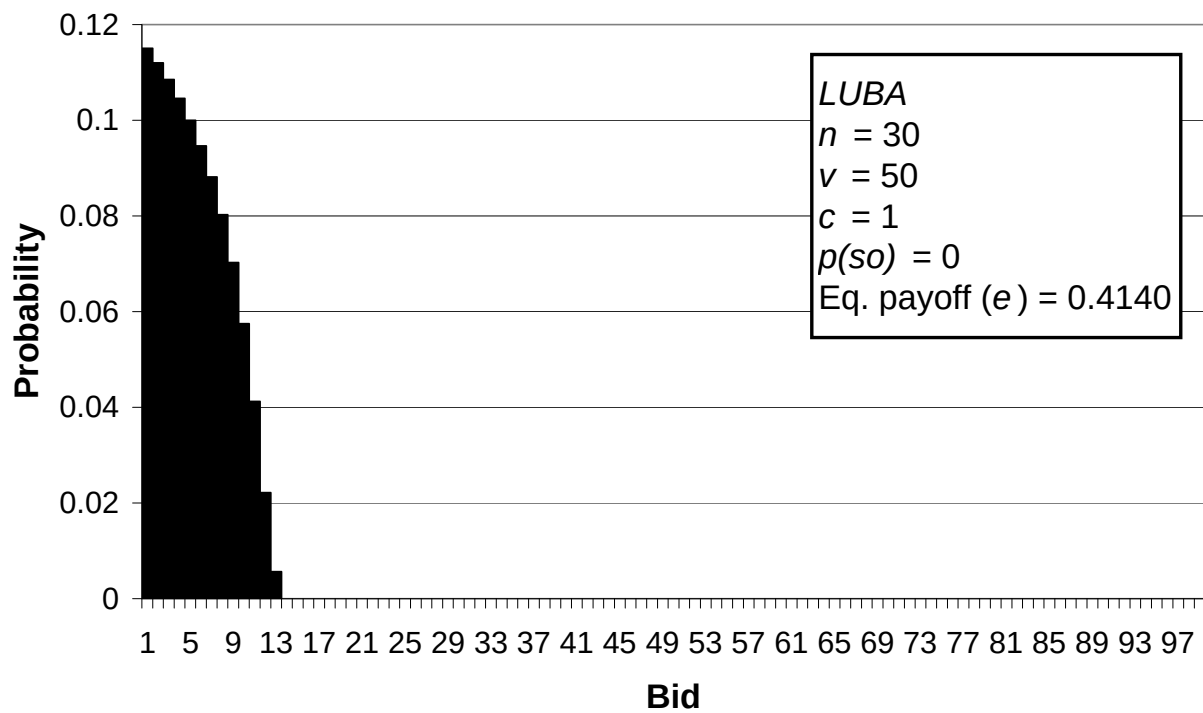
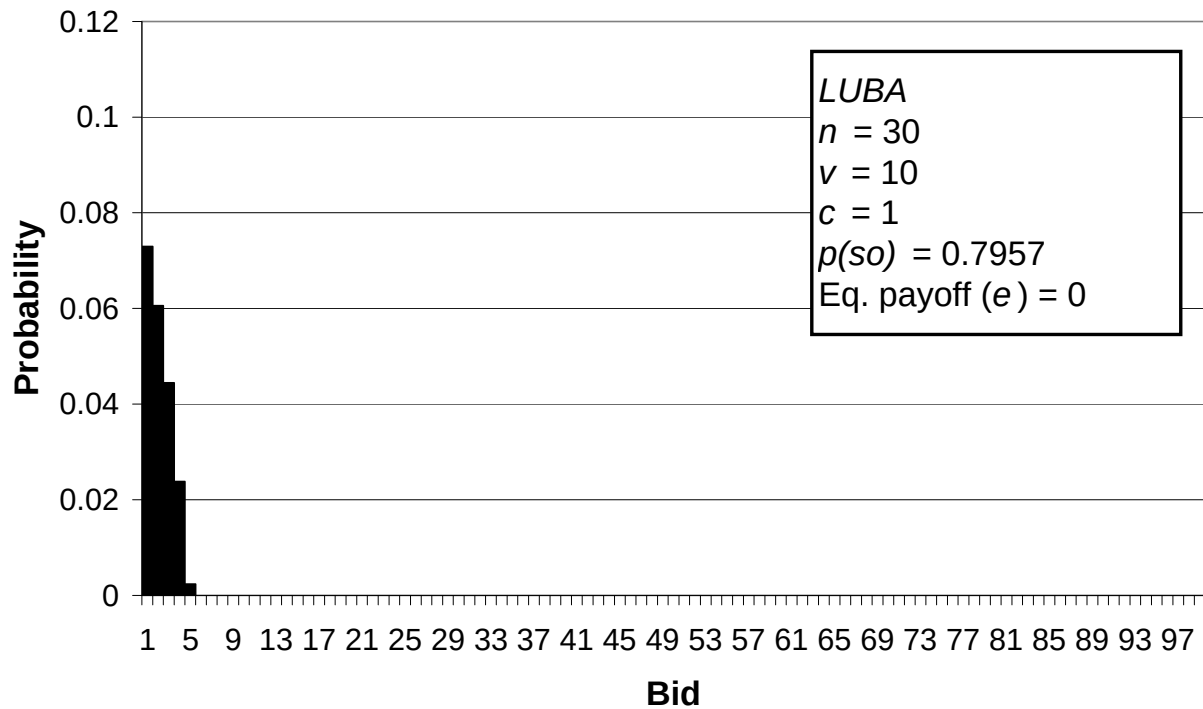


Figure 3: Symmetric mixed-strategy equilibria for the LUBA game for  $v=10, 50$ , and  $90$ ,  $n = 30$ , and  $c = 1$ .



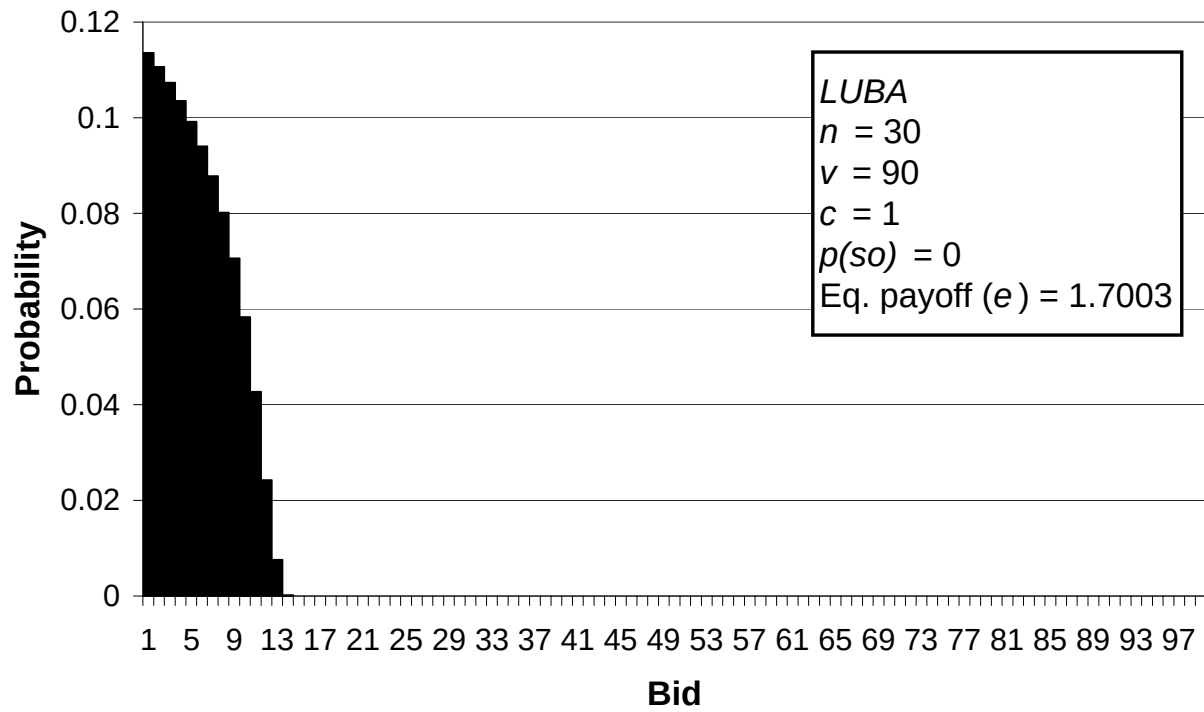
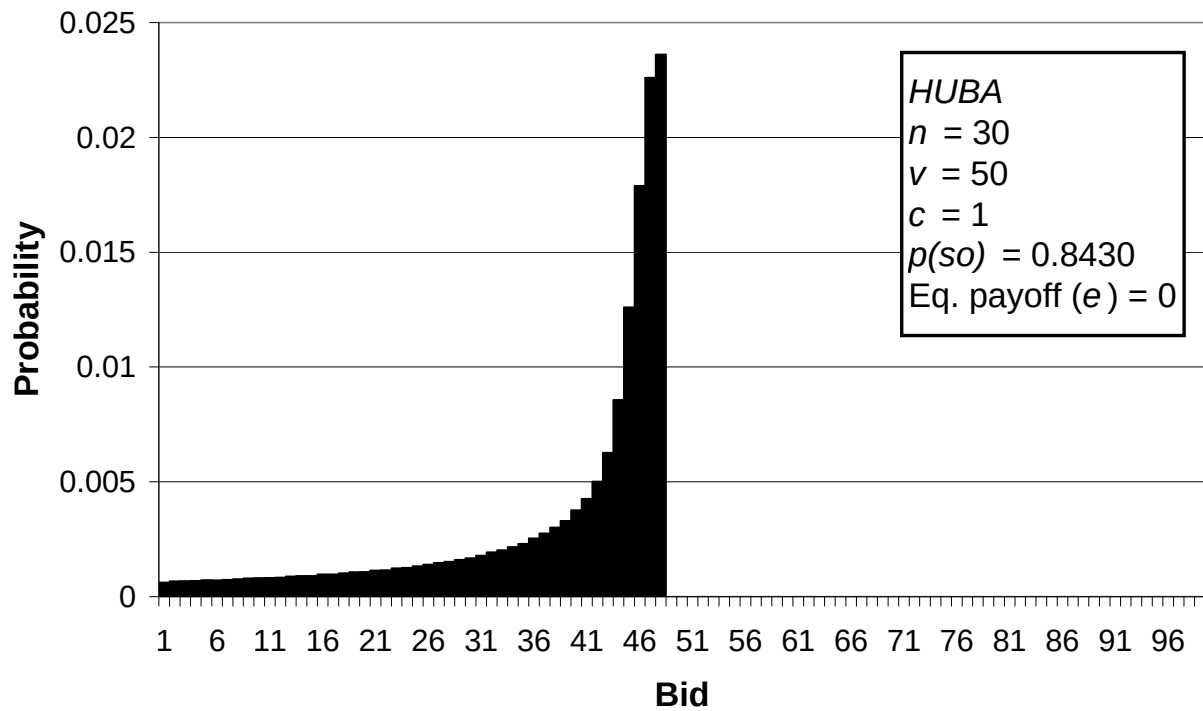
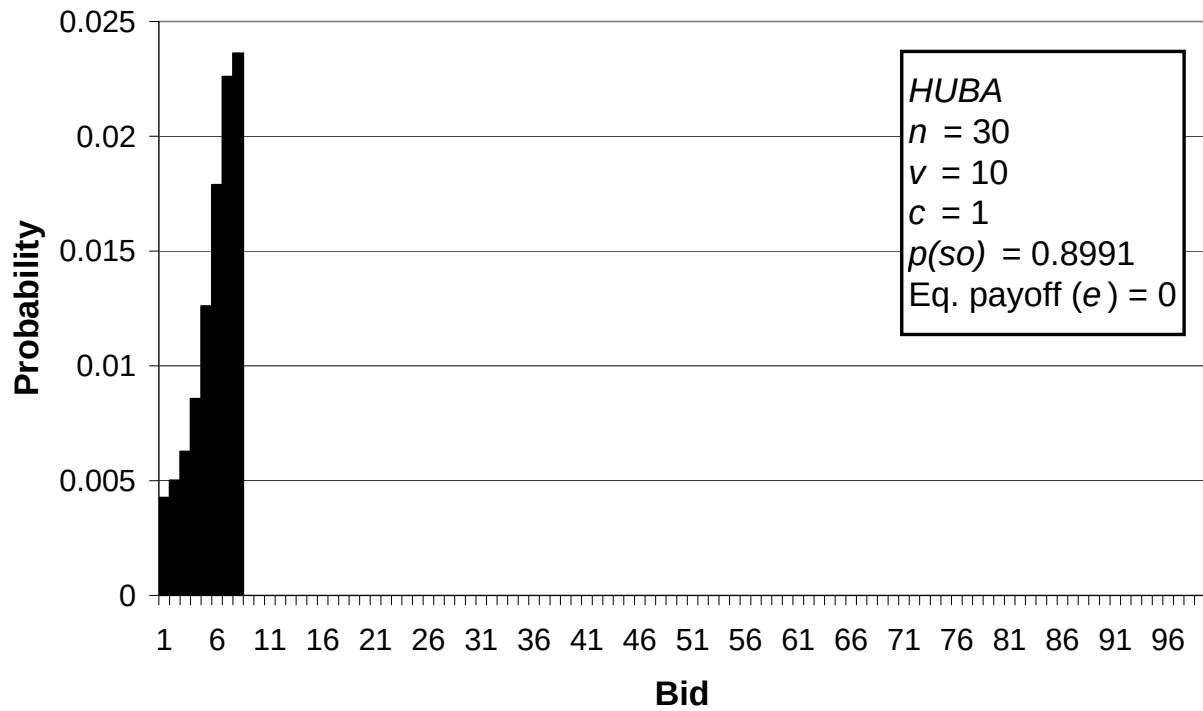


Figure 4: Symmetric mixed-strategy equilibria for the HUBA game for  $v=10, 50$ , and  $90$ ,  $n = 30$ , and  $c = 1$ .



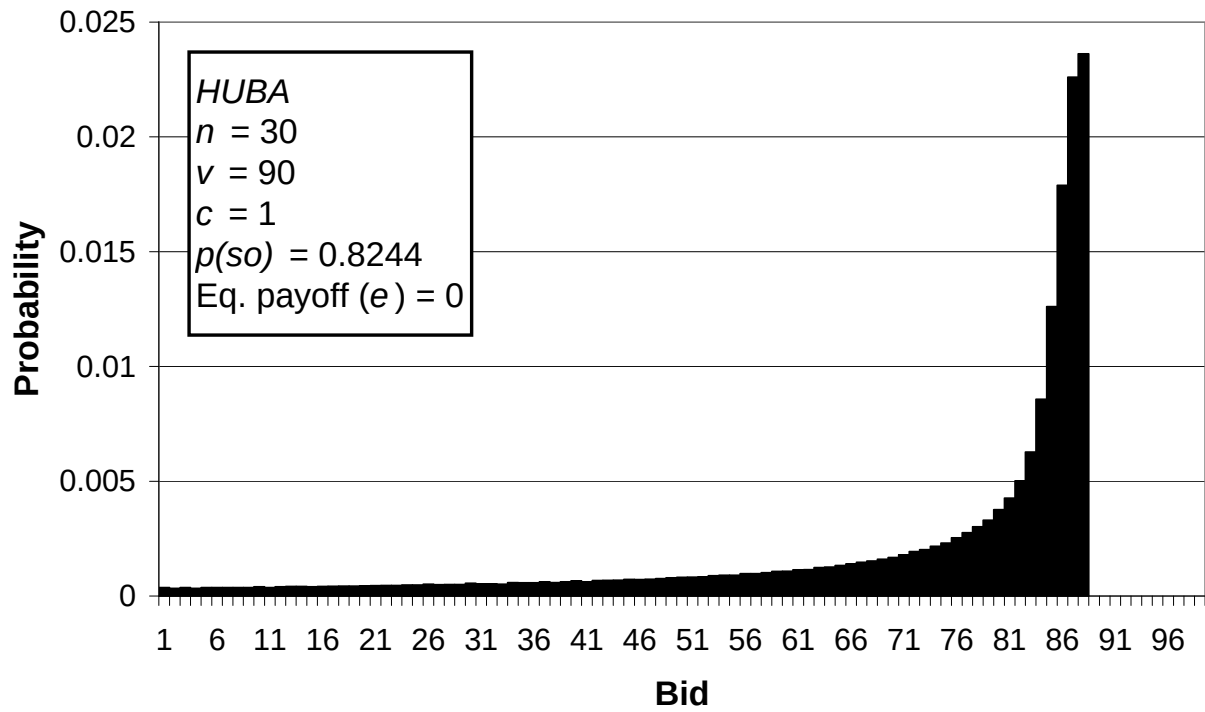
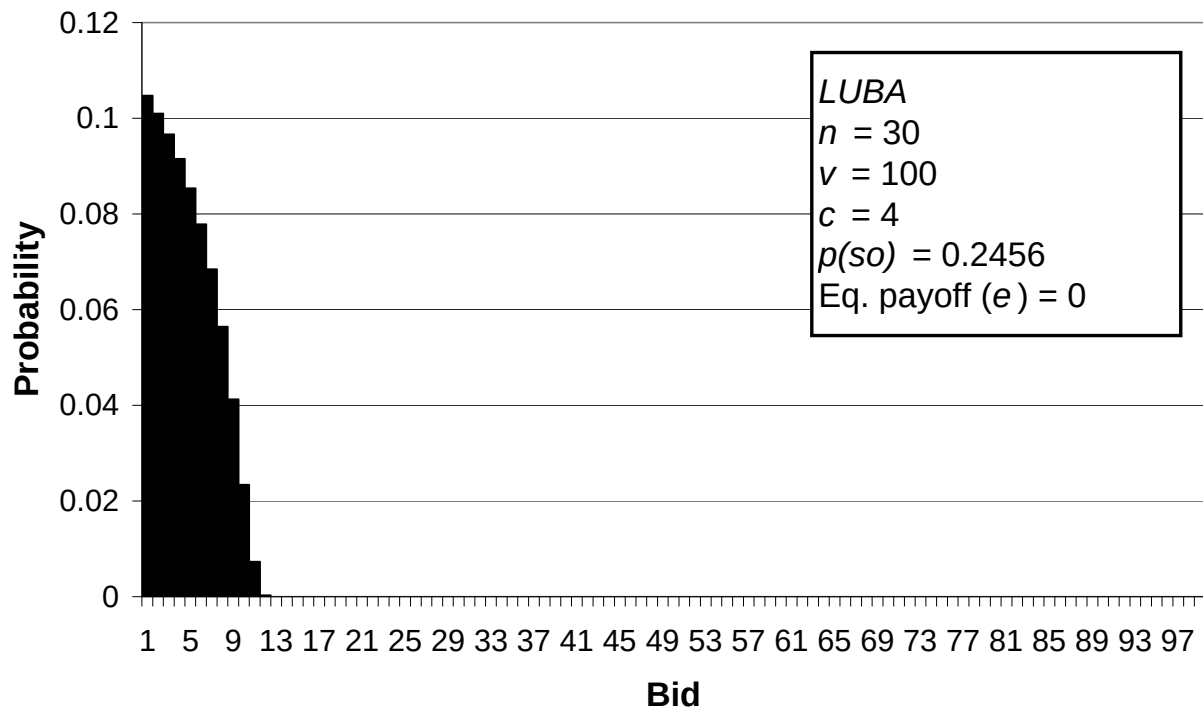
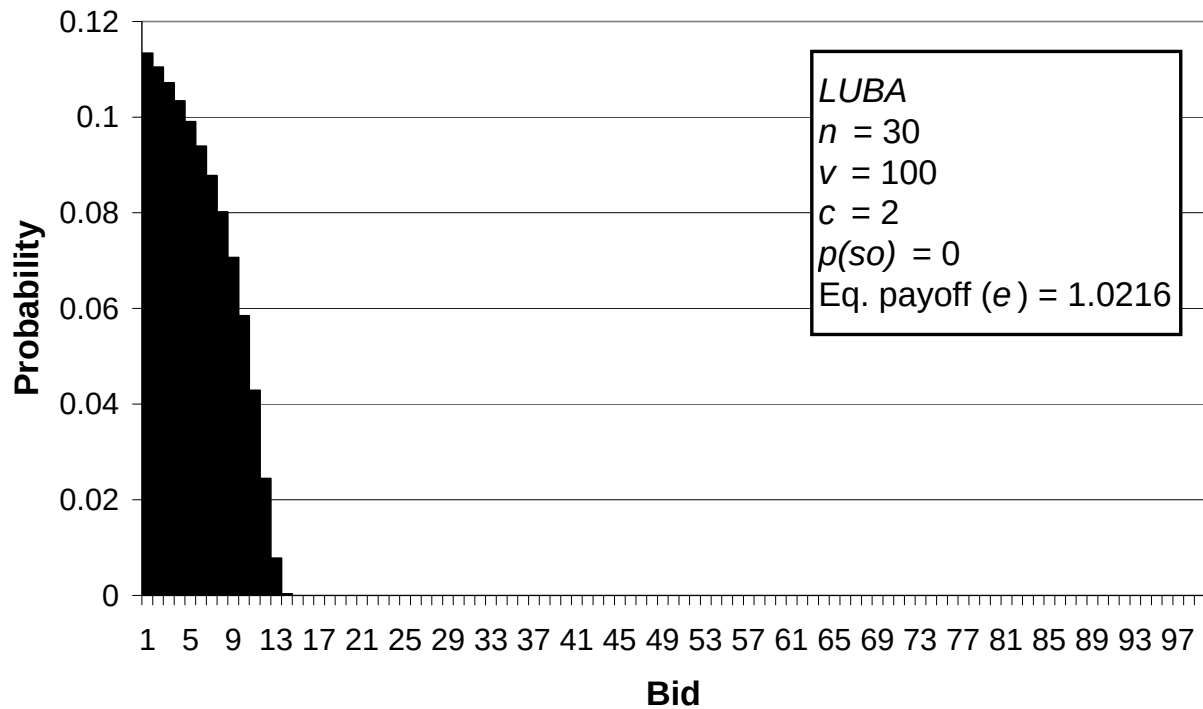


Figure 5: Symmetric mixed-strategy equilibria for the LUBA game for  $c=2, 4$ , and  $7$ ,  $n = 30$ , and  $v = 100$ .



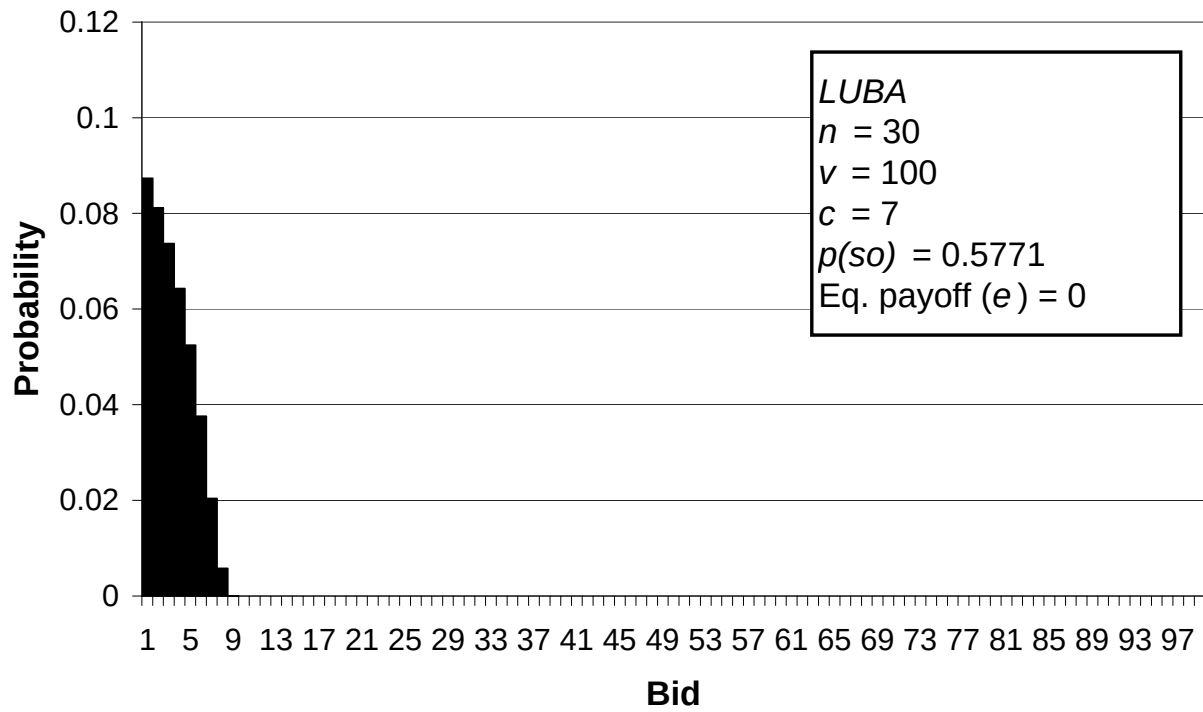


Figure 6: Symmetric mixed-strategy equilibria for the HUBA game for  $c=2, 4$ , and  $7$ ,  $n = 30$ , and  $v = 100$ .

